Oriented Quantum Algebras and Coalgebras, Invariants of Oriented 1–1 Tangles, Knots and Links

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This paper is the third in a series on oriented quantum algebras, structures related to them, and regular isotopy invariants associated with them. There is always a regular isotopy invariant of oriented 1–1 tangles associated to an oriented quantum algebra. Regular isotopy invariants of oriented knots and links can be constructed from oriented quantum algebras with a bit more structure. These are the twist oriented quantum algebras and they account for a very large number of the known regular isotopy invariants of oriented knots and links.

In this paper we study oriented quantum coalgebras which are structures closely related to oriented quantum algebras. We study the relationship between oriented quantum coalgebras and oriented quantum algebras and the relationship between oriented quantum coalgebras and quantum coalgebras. We show that there are regular isotopy invariants of oriented 1–1 tangles and

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of oriented knots and links associated to oriented and twist oriented quantum coalgebras respectively. There are many parallels between the theory of oriented quantum coalgebras and the theory of quantum coalgebras; the latter are introduced and studied in [11].

In the first paper [9] of this series the notion of oriented quantum algebra is introduced in the context of a very natural diagrammatic formalism. In the second [8] basic properties of oriented quantum algebras are described. Several examples of oriented quantum algebras are given, one of which is a paramaterized family which accounts for the Jones and HOMFLY polynomials.

This paper is organized as follows. In Section 1 we review most of the coalgebra prerequisites for this paper. Not much is required. The theory of coalgebras needed for this paper is more than adequately covered in any of [12, 13, 17]. In Section 2 we review the notions of quantum algebra, quantum coalgebra and their oriented counterparts. We also recall examples of oriented quantum algebras described in [9, 10]. We explore duality relationships between oriented algebra and coalgebra structures.

Section 3 is devoted to the relationship between oriented quantum algebras and quantum algebras. We have shown that a quantum algebra has an oriented quantum algebra structure. Here we show how to associate a quantum algebra to an oriented quantum algebra in a very natural way. In Section 4 we prove some general results on oriented quantum coalgebras. Section 5 is the coalgebra version of Section 3.

In Section 6 we define a function from the set of oriented 1–1 tangle diagrams with respect to a vertical to the dual algebra of an oriented quantum coalgebra and prove that this function determines a regular isotopy invariant of oriented 1–1 tangles. In Section 7 we show that the invariant of Section 6 is no better than the writhe when the oriented quantum coalgebra is cocommutative. One would expect this to be the case since the invariant of oriented 1–1 diagrams constructed from a commutative oriented quantum algebra has the same property.

In Section 8 the construction of the invariant of oriented 1–1 tangles described in Section 6 is used to give an invariant of oriented knots and links when the oriented quantum coalgebra is a twist oriented quantum coalgebra. The invariant for knots and links is a scalar.

This paper and some of its results were described in the survey paper [16]. Throughout k is a field and $k^\star$ will denote the set of non-zero elements

of k.

1 Preliminaries

For vector spaces U and V over k we will denote the tensor product $U \otimes_k V$ by $U \otimes V$, the identity map of V by 1_V and the linear dual $\mathrm{Hom}_k(V, k)$ of V by V^*. If T is a linear endomorphism of V then an element $v \in V$ is *T-invariant* if $T(v) = v$. If A is an algebra over k we shall let 1_A also denote the unit of k. Then meaning 1_V should always be clear from context.

We will usually denote a coalgebra (C, Δ, ϵ) over k by C and we will follow the convention of writing the coproduct $\Delta(c)$ symbolically as $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for all $c \in C$. This way of writing $\Delta(c)$ is a variation of the Heyneman–Sweedler notation. The *opposite coalgebra*, which we denote by C^{cop}, is $(C, \Delta^{cop}, \epsilon)$, where $\Delta^{cop}(c) = c_{(2)} \otimes c_{(1)}$ for all $c \in C$. An element $c \in C$ is said to be *cocommutative* if $\Delta(c) = c_{(2)} \otimes c_{(1)} = \Delta^{cop}(c)$. The coalgebra C is said to be a *cocommutative coalgebra* if all of its elements are cocommutative, or equivalently if $C = C^{cop}$.

Set $\Delta^{(1)} = \Delta$ and define $\Delta^{(n)} : C \longrightarrow C \otimes \cdots \otimes C$ ($n+1$ summands) for $n > 1$ inductively by $\Delta^{(n)} = (\Delta \otimes 1_C \otimes \cdots \otimes 1_C) \circ \Delta^{(n-1)}$. We generalize our notation for the coproduct and write $\Delta^{(n-1)}(c) = c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n)}$ for all $c \in C$.

A coalgebra which the reader will encounter several times in this paper is the *comatrix coalgebra* $\mathrm{C}_n(k)$ which is defined for all $n \geq 1$. As a k-vector space $\mathrm{C}_n(k)$ has basis $\{e^\imath_\jmath\}_{1 \leq \imath, \jmath \leq n}$. The coproduct and the counit for $\mathrm{C}_n(k)$ are determined by

$$\Delta(e^\imath_\jmath) = \sum_{\ell=1}^{n} e^\imath_\ell \otimes e^\ell_\jmath \quad \text{and} \quad \epsilon(e^\imath_\jmath) = \delta^\imath_\jmath$$

respectively for all $1 \leq \imath, \jmath \leq n$.

We usually denote an algebra (A, m, η) over k by A. The *opposite algebra* is the k-algebra (A, m^{op}, η) whose product is defined by $m^{op}(a \otimes b) = m(b \otimes a) = ba$ for all $a, b \in A$. We denote the opposite algebra by A^{op}. For $n \geq 1$ let $\mathrm{M}_n(k)$ the algebra of all $n \times n$ matrices with entries in k and let $\{E^\imath_\jmath\}_{1 \leq \imath, \jmath \leq n}$ be the standard basis for $\mathrm{M}_n(k)$. In our notation $E^\imath_\jmath E^\ell_m = \delta^\ell_\jmath E^\imath_m$ for all $1 \leq \imath, \jmath, \ell, m \leq n$.

Let C be a coalgebra over k. Then C^* is an algebra over k, called the *dual algebra*, or *algebra dual to* C, whose product is determined by

$$c^*d^*(c) = c^*(c_{(1)})d^*(c_{(2)})$$

for all $c^*, d^* \in C^*$ and $c \in C$ and whose unit is ϵ. Note that C is a C^*-bimodule under the left and right actions

$$c^* \rightharpoonup c = c_{(1)}c^*(c_{(2)}) \quad \text{and} \quad c \leftharpoonup c^* = c^*(c_{(1)})c_{(2)}$$

for all $c^* \in C^*$ and $c \in C$.

Now suppose that A is an algebra over k. Then the subspace A^o of A^* consisting of all functionals which vanish on a cofinite ideal of A is a coalgebra over k. For $a^o \in A^o$ the coproduct $\Delta(a^o) = \sum_{\imath=1}^r a^o_\imath \otimes b^o_\imath$ is determined by

$$a^o(ab) = \sum_{\imath=1}^r a^o_\imath(a)b^o_\imath(b)$$

for all $a, b \in A$ and the counit is given by $\epsilon(a^o) = a^o(1)$. If $f : A \longrightarrow B$ is an algebra map then the restriction f^o of the transpose map $f^* : B^* \longrightarrow A^*$ determines a coalgebra map $f^o : B^o \longrightarrow A^o$. Note that $A^o = A^*$ when A is finite-dimensional. Observe that $\mathrm{C}_n(k) \simeq \mathrm{M}_n(k)^*$ as coalgebras and that $\{e^\imath_\jmath\}_{1 \leq \imath,\jmath \leq n}$ can be identified with the basis for $\mathrm{C}_n(k)$ dual to the standard basis $\{E^\imath_\jmath\}_{1 \leq \imath,\jmath \leq n}$ for $\mathrm{M}_n(k)$.

Let V be a vector space over k and suppose that $b : V \times V \longrightarrow k$ is a bilinear form. We define $b_{(\ell)}, b_{(r)} : V \longrightarrow V^*$ by $b_{(\ell)}(u)(v) = b(u, v) = b_{(r)}(v)(u)$ for all $u, v \in V$. For $\rho \in V \otimes V$ we define a bilinear form $b_\rho : V^* \otimes V^* \longrightarrow k$ by $b_\rho(u^*, v^*) = (u^* \otimes v^*)(\rho)$ for all $u^*, v^* \in V^*$.

Now suppose that C, D are coalgebras over k and let $b, b' : C \times D \longrightarrow k$ be bilinear forms. Then b' is an *inverse for* b if

$$b'(c_{(1)}, d_{(1)})b(c_{(2)}, d_{(2)}) = \epsilon(c)\epsilon(d) = b(c_{(1)}, d_{(1)})b'(c_{(2)}, d_{(2)})$$

for all $c \in C$ and $d \in D$. The bilinear form b has at most one inverse which we denote b^{-1} when it exists.

2 Oriented Quantum Algebras and Coalgebras, Definitions and Examples

In this section we recall the definition of quantum algebra, oriented quantum algebra, quantum coalgebra and oriented quantum coalgebra and list some

examples of these structures which are found in [9] and [10]. We consider duality relations between these algebra and coalgebra structures.

An important component of the definition of quantum algebra or oriented quantum algebra is a solution to a the quantum Yang–Baxter equation. Let A be an algebra over the field k, $\rho \in A \otimes A$ and write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$. For $1 \leq \imath < \jmath \leq 3$ let $\rho_{\imath\,\jmath} \in A \otimes A \otimes A$ be defined by

$$\rho_{1\,2} = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath \otimes 1, \quad \rho_{1\,3} = \sum_{\imath=1}^{r} a_\imath \otimes 1 \otimes b_\imath \quad \text{and} \quad \rho_{2\,3} = \sum_{\imath=1}^{r} 1 \otimes a_\imath \otimes b_\imath.$$

The quantum Yang–Baxter equation for ρ is $\rho_{1\,2}\rho_{1\,3}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,3}\rho_{1\,2}$.

The notion of quantum algebra is defined in [4]. A *quantum algebra* over the field k is a triple (A, ρ, s), where A is an algebra over k, $\rho \in A \otimes A$ is invertible and $s : A \longrightarrow A^{op}$ is an algebra isomorphism, such that

(QA.1) $\rho^{-1} = (s \otimes 1_A)(\rho)$,

(QA.2) $\rho = (s \otimes s)(\rho)$ and

(QA.3) $\rho_{1\,2}\rho_{1\,3}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,3}\rho_{1\,2}$.

Suppose that (A', ρ', s') is a quantum algebra over k also. A *morphism of quantum algebras* $f : (A, \rho, s) \longrightarrow (A', \rho', s')$ is an algebra map $f : A \longrightarrow A'$ which satisfies $\rho' = (f \otimes f)(\rho)$ and $s' \circ f = f \circ s$. Quantum algebras together with their morphisms under composition form a monoidal category. The reader is referred to [11, Section 3] at this point.

Our first example of a quantum algebra accounts for the Jones polynomial when $k = \mathbb{C}$ is the field of complex numbers. See [4, page 580] and also [10, Section 3].

Example 1 *Let $q \in k^\star$. Then $(\mathrm{M}_2(k), \rho, s)$ is a quantum algebra over the field k, where*

$$\rho = q^{-1}(E^1_1 \otimes E^1_1 + E^2_2 \otimes E^2_2) + q(E^1_1 \otimes E^2_2 + E^2_2 \otimes E^1_1) + (q^{-1} - q^3)E^1_2 \otimes E^2_1$$

and $s(x) = Mx^tM^{-1}$ for all $x \in \mathrm{M}_2(k)$, where $M = \begin{pmatrix} 0 & q \\ -q^{-1} & 0 \end{pmatrix}$.

Finite-dimensional quasitriangular Hopf algebras account for a large class of quantum algebras.

Example 2 *Let* (A, ρ) *be a quasitriangular Hopf algebra with antipode* s *over the field* k. *Then* (A, ρ, s) *is a quantum algebra over* k.

The notion of oriented quantum algebra is introduced in [9, Section 1]. An *oriented quantum algebra* over the field k is a quadruple $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$, where A is an algebra over k, $\rho \in A \otimes A$ is invertible and $t_{\mathsf{d}}, t_{\mathsf{u}}$ are commuting algebra automorphisms of A, such that

(qa.1) $(t_{\mathsf{d}} \otimes 1_A)(\rho^{-1})$ and $(1_A \otimes t_{\mathsf{u}})(\rho)$ are inverses in $A \otimes A^{op}$,

(qa.2) $\rho = (t_{\mathsf{d}} \otimes t_{\mathsf{d}})(\rho) = (t_{\mathsf{u}} \otimes t_{\mathsf{u}})(\rho)$ and

(qa.3) $\rho_{1\,2}\rho_{1\,3}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,3}\rho_{1\,2}$.

An oriented quantum algebra $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is *standard* if $t_{\mathsf{d}} = 1_A$ and is *balanced* if $t_{\mathsf{d}} = t_{\mathsf{u}}$. In the balanced case we write (A, ρ, t) for (A, ρ, t, t).

Suppose that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ and $(A, \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ are oriented quantum algebras over k. A *morphism of oriented quantum algebras* $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ is an algebra map $f : A \longrightarrow A'$ which satisfies $\rho' = (f \otimes f)(\rho)$, $t'_{\mathsf{d}} \circ f = f \circ t_{\mathsf{d}}$ and $t'_{\mathsf{u}} \circ f = f \circ t_{\mathsf{u}}$. Oriented quantum algebras together with their morphisms under composition form a monoidal category.

Theorem 1 of [9] accounts for an extensive family of examples of balanced oriented quantum algebras.

Example 3 *Let* $n \geq 2$ *and* $x, \mathit{bc} \in k^\star$. *Then* $(\mathrm{M}_n(k), \rho, t)$ *is a balanced oriented quantum algebra over* k *where* $\rho = \sum_{\imath,\jmath,\ell,m=1}^{n} \rho_{\jmath\, m}^{\imath\, \ell} E_\jmath^\imath \otimes E_m^\ell$ *satisfies*

a) $\rho_{\jmath\, m}^{\imath\, \ell} = 0$ *unless* $\{\imath, \ell\} = \{\jmath, m\}$,

b) $\rho_{\imath\, \jmath}^{\imath\, \jmath} \neq 0$ *for all* $1 \leq \imath, \jmath \leq n$,

c) $\rho_{\jmath\, \imath}^{\imath\, \jmath} = x = \rho_{\imath\, \imath}^{\imath\, \imath} - \mathit{bc}/\rho_{\imath\, \imath}^{\imath\, \imath}$ *and* $\rho_{\imath\, \jmath}^{\jmath\, \imath} = 0$ *for all* $1 \leq \imath < \jmath \leq n$,

d) $\rho_{\imath\, \jmath}^{\imath\, \jmath}\rho_{\jmath\, \imath}^{\jmath\, \imath} = \mathit{bc}$ *for all* $1 \leq \imath < \jmath \leq n$,

e) *for all* $1 \leq \imath, \jmath \leq n$ *either* $\rho_{\imath\, \imath}^{\imath\, \imath} = \rho_{\jmath\, \jmath}^{\jmath\, \jmath}$ *or* $\rho_{\imath\, \imath}^{\imath\, \imath}\rho_{\jmath\, \jmath}^{\jmath\, \jmath} = -\mathit{bc}$

and $t(E_\jmath^\imath) = (\omega_\imath/\omega_\jmath)E_\jmath^\imath$ *for all* $1 \leq \imath, \jmath \leq n$, *where* $\omega_1, \ldots, \omega_n \in k^\star$ *satisfy*

$$\omega_\imath^2 = \left(\frac{\rho_{1\,1}^{1\,1}\rho_{\imath\,\imath}^{\imath\,\imath}}{\mathit{bc}}\right)\left(\prod_{1<\jmath<\imath}\frac{(\rho_{\jmath\,\jmath}^{\jmath\,\jmath})^2}{\mathit{bc}}\right)\omega_1^2$$

for all $1 < \imath \leq n$.

Let $k = \mathbb{C}$ be the field of complex numbers and suppose that $q \in \mathbb{C}^*$ is transcendental over the subfield of rational numbers. When $b\!c = q^2$, $x = q^{-1} - q^3$, $\rho^{\imath\imath}_{\imath\imath} = q^{-1}$ for all $1 \leq \imath \leq n$ and $\rho^{\imath\jmath}_{\imath\jmath} = q^2$ whenever $1 \leq \imath, \jmath \leq n$ are distinct, then Example 3 accounts for the HOMFLY polynomial.

A quantum algebra always has an oriented quantum algebra structure by virtue of [8, Propositions 1 and 2].

Example 4 *If (A, ρ, s) is a quantum algebra over the field k then $(A, \rho, s^{-2}, 1_A)$ and $(A, \rho, 1_A, s^{-2})$ are oriented quantum algebras over k.*

A quantum algebra (A, ρ, s) over k may have no oriented quantum algebra structures of the type $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ except those mentioned in the preceding example; see Example 4 of [8]. A balanced oriented quantum algebra (A, ρ, t) over k may not have a quantum algebra structure of the type (A, ρ, s); see Example 3 of [8].

Balanced oriented quantum algebras arise in very natural ways.

Example 5 *Let (A, ρ) be a finite-dimensional quasitriangular Hopf algebra over the field k and suppose that t is a Hopf algebra automorphism of A which satisfies $\rho = (t \otimes t)(\rho)$ and $t^2 = s^{-2}$. Then (A, ρ, t) is a balanced oriented quantum algebra.*

Very important examples of a finite-dimensional quasitriangular Hopf algebras over k are the quantum doubles $(D(A), \boldsymbol{\rho})$ of finite-dimensional Hopf algebras A with antipode s over k. We write $D(A) = A^* \otimes A$ as a vector space.

Example 6 *Let A be a finite-dimensional Hopf algebra over k and suppose that t is a Hopf algebra automorphism of A which satisfies $t^2 = s^{-2}$. Then $(D(A), \boldsymbol{\rho}, T)$ is a balanced oriented quantum algebra over k, where $T = (t^{-1})^* \otimes t$.*

For details concerning these two examples see [9, Corollary 2] and the discussion preceding it.

We now turn to quantum coalgebras and oriented quantum coalgebras. The notion of quantum coalgebra was introduced in [11, Section 4]. Strict quantum coalgebras form an important class of quantum coalgebras.

A *strict quantum coalgebra over k* is a triple (C, b, S), where C is a coalgebra over k, $b : C \times C \longrightarrow k$ is an invertible bilinear form and $S : C \longrightarrow C^{cop}$ is a coalgebra isomorphism, such that

(QC.1) $b^{-1}(c,d) = b(S(c),d)$,
(QC.2) $b(c,d) = b(S(c),S(d))$ and
(QC.3) $b(c_{(1)},d_{(1)})b(c_{(2)},e_{(1)})b(d_{(2)},e_{(2)}) = b(c_{(2)},d_{(2)})b(c_{(1)},e_{(2)})b(d_{(1)},e_{(1)})$

for all $c,d,e \in C$. A *quantum coalgebra over* k is a triple (C,b,S), where C is a coalgebra over k, $b : C \times C \longrightarrow k$ is an invertible bilinear form, $S : C \longrightarrow C^{cop}$ is a coalgebra isomorphism of C with respect to b, such that (QC.1)–(QC.3) hold. That S is a coalgebra isomorphism with respect to b means S is a linear isomorphism which satisfies $\epsilon \circ S = \epsilon$,

$$b(S(c_{(1)}),d)b(S(c_{(2)}),e) = b(S(c)_{(2)},d)b(S(c)_{(1)},e) \text{ and}$$

$$b(d,S(c_{(1)}))b(e,S(c_{(2)})) = b(d,S(c)_{(2)})b(e,S(c)_{(1)})$$

for all $c,d,e \in C$.

A *morphism of quantum coalgebras* $f : (C,b,S) \longrightarrow (C',b',S')$ is a coalgebra map $f : C \longrightarrow C'$ which satisfies $b(c,d) = b'(f(c),f(d))$ for all $c,d \in C$ and $S' \circ f = f \circ S$. Quantum coalgebras over k together with their morphisms under composition form a monoidal category; the strict quantum coalgebras over k form a subcategory of this category.

The notions of quantum algebra and strict quantum coalgebra are dual as was remarked in [11, Section 4.1]. More formally,

Proposition 1 *Let A be a finite-dimensional algebra over k, let $\rho \in A \otimes A$ and suppose that s is a linear automorphism of A. Let A^* be the dual coalgebra of A. Then the following are equivalent:*

a) *(A,ρ,s) is a quantum algebra over k.*

b) *(A^*,b_ρ,s^*) is a strict quantum coalgebra over k.*

□

A little more can be squeezed from a proof of the proposition.

Corollary 1 *Suppose that (A,ρ,s) is any quantum algebra over k. Then (A^o,b,s^o) is a strict quantum coalgebra over k, where $b(a^o,b^o) = (a^o \otimes b^o)(\rho)$ for all $a^o,b^o \in A^o$.*

□

The strict quantum coalgebra (A^o, b, s^o) of Corollary 1 is called the *dual quantum coalgebra of* (A, ρ, s). The dual quantum coalgebra of the quantum algebra of Example 1 is a basic example of a (strict) quantum coalgebra.

Example 7 *Let* $q \in k^\star$. *Then* $(\mathrm{C}_2(k), b, S)$ *is a quantum coalgebra over* k *where*

$$b(e^1_1, e^1_1) = q^{-1} = b(e^2_2, e^2_2), \quad b(e^1_1, e^2_2) = q = b(e^2_2, e^1_1), \quad b(e^1_2, e^2_1) = q^{-1} - q^3$$

and $b(e^\imath_\jmath, e^\ell_m) = 0$ *otherwise, and*

$$S(e^1_1) = e^2_2, \quad S(e^2_2) = e^1_1, \quad S(e^1_2) = -q^2 e^1_2 \text{ and } S(e^2_1) = -q^{-2} e^2_1.$$

Also see [11, Section 8].

Just as finite-dimensional quasitriangular Hopf algebras give rise to quantum algebras, it is easy to see, following the discussion of [12, Section 7.3] for example, that:

Example 8 *Let* (A, β) *be a coquasitriangular Hopf algebra with antipode* s *over the field* k. *Then* (A, β, s) *is a quantum coalgebra over* k.

The notion of oriented quantum coalgebra is introduced in [8, Section 4]. Strict oriented quantum coalgebras form an important class of oriented quantum coalgebras. A *strict oriented quantum coalgebra over* k is a quadruple $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$, where C is a coalgebra over k, $b : C \times C \longrightarrow k$ is an invertible bilinear form and $T_{\mathsf{d}}, T_{\mathsf{u}}$ are commuting coalgebra automorphisms of C, such that

(qc.1) $b(c_{(1)}, T_{\mathsf{u}}(d_{(2)}))b^{-1}(T_{\mathsf{d}}(c_{(2)}), d_{(1)}) = \epsilon(c)\epsilon(d)$ and
$b^{-1}(T_{\mathsf{d}}(c_{(1)}), d_{(2)})b(c_{(2)}, T_{\mathsf{u}}(d_{(1)})) = \epsilon(c)\epsilon(d)$,

(qc.2) $b(c, d) = b(T_{\mathsf{d}}(c), T_{\mathsf{d}}(d)) = b(T_{\mathsf{u}}(c), T_{\mathsf{u}}(d))$ and

(qc.3) $b(c_{(1)}, d_{(1)})b(c_{(2)}, e_{(1)})b(d_{(2)}, e_{(2)}) = b(c_{(2)}, d_{(2)})b(c_{(1)}, e_{(2)})b(d_{(1)}, e_{(1)})$

for all $c, d, e \in C$. An *oriented quantum coalgebra over* k is a quadruple $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$, where C is a coalgebra over k, $b : C \times C \longrightarrow k$ is an invertible bilinear form and T is a coalgebra automorphism of C with respect to $\{b, b^{-1}\}$, such that (qc.1)–(qc.3) hold. Generally if C, D are coalgebras over k and $\mathcal{S}$

is a set of bilinear forms $b : D{\times}D \longrightarrow k$, then a linear map (respectively isomorphism) $T : C \longrightarrow D$ is a *coalgebra map* (respectively *isomorphism*) *with respect to* $\mathcal{S}$ if

$$b(T(c_{(1)}), d)b'(T(c_{(2)}), e) = b(T(c)_{(1)}, d)b'(T(c)_{(2)}, e)$$

and

$$b(d, T(c_{(1)}))b'(e, T(c_{(2)})) = b(d, T(c)_{(1)})b'(e, T(c)_{(2)})$$

for all $b, b' \in \mathcal{S}$, $c \in C$ and $d, e \in D$. When $C = D$ and T is a coalgebra isomorphism with respect to $\mathcal{S}$ then T is called a *coalgebra automorphism of* C *with respect to* $\mathcal{S}$.

An oriented quantum coalgebra $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is *standard* if $T_{\mathsf{d}} = 1_C$ and is *balanced* if $T_{\mathsf{d}} = T_{\mathsf{u}}$. In the balanced case we write (C, b, T) for (C, b, T, T). A *morphism of oriented quantum coalgebras* $f : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ is a coalgebra map $f : C \longrightarrow C'$ which satisfies $b(c, d) = b'((f(c), f(d))$ for all $c, d \in C$ and $T'_{\mathsf{d}}{\circ}f = f{\circ}T_{\mathsf{d}}$, $T'_{\mathsf{u}}{\circ}f = f{\circ}T_{\mathsf{u}}$. Oriented quantum coalgebras together with their morphisms under composition form a monoidal category.

As remarked in [9, Section 3], the notions of oriented quantum algebra and strict oriented quantum coalgebra are dual. We state here more formally:

Proposition 2 *Suppose that A is a finite-dimensional algebra over k, $\rho \in A{\otimes}A$ and $t_{\mathsf{d}}, t_{\mathsf{u}}$ are commuting linear automorphisms of A. Let A^* be the dual coalgebra of A. Then the following are equivalent:*

a) $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *is an oriented quantum algebra over k.*

b) $(A^*, b_\rho, t^*_{\mathsf{d}}, t^*_{\mathsf{u}})$ *is a strict oriented quantum coalgebra over k.*

□

Moreover:

Corollary 2 *Suppose that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is any oriented quantum algebra over k. Then $(A^o, b, t^o_{\mathsf{d}}, t^o_{\mathsf{u}})$ is a strict oriented quantum coalgebra over k, where $b(a^o, b^o) = (a^o{\otimes}b^o)(\rho)$ for all $a^o, b^o \in A^o$.*

□

The strict oriented quantum coalgebra $(A^o, b, t^o_{\mathsf{d}}, t^o_{\mathsf{d}})$ of Corollary 2 is called the *dual oriented quantum coalgebra of* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$. The duals of the strict quantum algebras of Example 3 form a rather extensive family of balanced strict oriented quantum coalgebras.

Example 9 *Suppose $n \geq 2$ and $bc, x \in k^\star$. Suppose that $\{\rho^{\imath\,\ell}_{\jmath\,m}\}_{1\leq \imath,\jmath\leq n} \subseteq k$ and $\{\omega_\imath\}_{1\leq\imath\leq n} \subseteq k^\star$ satisfy conditions a)–d) and e) respectively of Example 3. Then $(\mathrm{C}_n(k), b, T)$ is a strict balanced oriented quantum coalgebra over k, where $b(e^\imath_\jmath, e^\ell_m) = \rho^{\imath\,\ell}_{\jmath\,m}$ for all $1 \leq \imath, \jmath, \ell, m \leq n$ and $T(e^\imath_\jmath) = (\omega_\imath/\omega_\jmath)e^\imath_\jmath$ for all $1 \leq \imath, \jmath \leq n$.*

Observe that the quantum coalgebra $(\mathrm{C}_2(k), b, S)$ of Example 7 has a strict oriented balanced quantum coalgebra structure $(\mathrm{C}_2(k), b, T)$ which is a special case of the previous example with $\omega_1 = q^{-1}$ and $\omega_2 = -q$. Note that $S\circ T = T\circ S$ and $T^2 = S^{-2}$.

We end this section with a result on coalgebra automorphisms with respect to a set of bilinear forms which will be useful for the proof of Theorem 4 of Section 6.2. Let C be a coalgebra over k and suppose that $\mathcal{S}$ is a set of bilinear forms $b : C\times C \longrightarrow k$. The set of linear automorphisms T of C which satisfy (qc.2) for all $b \in \mathcal{S}$ is easily seen to be a subgroup of the multiplicative group of all linear automorphisms of C.

Lemma 1 *Let C be a coalgebra over the field k and suppose that $\mathcal{S}$ is a set of bilinear forms $b : C\times C \longrightarrow k$.*

a) *The set of coalgebra automorphisms T of C with respect to $\mathcal{S}$ which satisfy (qc.2) for all $b \in \mathcal{S}$ form a subgroup $\mathcal{G}(C, \mathcal{S})$ of the group of linear automorphisms of C under composition.*

b) *The equations*

$$b(T^{u+\ell}(c_{(1)}), d)b'(T^{v+\ell}(c_{(2)}), e) = b(T^u(T^\ell(c)_{(1)}), d)b'(T^v(T^\ell(c)_{(2)}), e)$$

and

$$b(d, T^{u+\ell}(c_{(1)}))b'(e, T^{v+\ell}(c_{(2)})) = b(d, T^u(T^\ell(c)_{(1)}))b'(e, T^v(T^\ell(c)_{(2)}))$$

hold for all $b, b' \in \mathcal{S}$, for all $T \in \mathcal{G}(C, \mathcal{S})$, for all integers u, v, ℓ and $c, d, e \in C$.

Proof: It is clear that the identity map of C lies in $\mathcal{G}(C, \mathcal{S})$. Suppose that $T, U \in \mathcal{G}(C, \mathcal{S})$. To complete the proof of part a) we need only show that

$T^{-1}\circ U \in \mathcal{G}(C,\mathcal{S})$. For all $b, b' \in S$ and $c,d,e \in C$ observe that

$$\begin{aligned}
&b(T^{-1}(U(c_{(1)})),d)b'(T^{-1}(U(c_{(2)})),e) \\
&= b(U(c_{(1)}),T(d))b'(U(c_{(2)}),T(e)) \\
&= b(U(c)_{(1)},T(d))b'(U(c)_{(2)},T(e)) \\
&= b(T(T^{-1}(U(c)))_{(1)},T(d))b'(T(T^{-1}(U(c)))_{(2)},T(e)) \\
&= b(T(T^{-1}(U(c))_{(1)}),T(d))b'(T(T^{-1}(U(c))_{(2)}),T(e)) \\
&= b(T^{-1}(U(c))_{(1)},d)b'(T^{-1}(U(c)_{(2)},e)
\end{aligned}$$

and likewise

$$b(d,T^{-1}(U(c_{(1)})))b'(e,T^{-1}(U(c_{(2)}))) = b(d,T^{-1}(U(c))_{(1)})b'(e,T^{-1}(U(c))_{(2)}).$$

Therefore $T^{-1}\circ U \in \mathcal{G}(C,\mathcal{S})$. Since $b(T^{u+\ell}(c),d) = b(T^{\ell}(c),T^{-u}(d))$ and $b(d,T^{u+\ell}(c)) = b(T^{-u}(d),T^{\ell}(c))$ and for all integers u,ℓ and $c,d,e \in C$, part b) follows from part a). $\Box$

3 A Basic Relationship Between Oriented and Unoriented Quantum Algebra Structures

Suppose that (A,ρ,s) is a quantum algebra over k. Then $(A,\rho,1_A,s^{-2})$ is a standard oriented quantum algebra over k by virtue of Example 4. We have seen that not every oriented quantum algebra is of this form in our discussion following Example 4. The main purpose of this section is to show that a quantum algebra can be associated with an oriented quantum algebra in a natural way.

Let $(A,\rho,1_A,t)$ be a standard oriented quantum algebra over k, let $\mathcal{A} = A\oplus A^{op}$ be the direct product of A and A^{op}, and let $\pi : \mathcal{A} \longrightarrow A$ be the projection onto the first factor. We will construct a quantum algebra $(\mathcal{A},\boldsymbol{\rho},\mathbf{s})$ on $\mathcal{A}$ such that $\pi : (\mathcal{A},\rho,1_{\mathcal{A}},\mathbf{s}^{-2}) \longrightarrow (A,\rho,1_A,t)$ is a morphism of oriented quantum algebras.

Let $\overline{(\ )}$ denote the linear involution of $\mathcal{A}$ which exchanges the direct summands of $\mathcal{A}$. Thus $\overline{a\oplus b} = b\oplus a$ for all $a,b \in A$. We regard A as a subspace of $\mathcal{A}$ by the identification $a = a\oplus 0$ for all $a \in A$. Therefore $\overline{a} = 0\oplus a$ and every element of $\mathcal{A}$ has a unique decomposition of the form $a + \overline{b}$ for some

$a, b \in A$. Observe that

$$\overline{(\overline{a})} = a, \quad \overline{ab} = \overline{b}\overline{a} \text{ and } a\overline{b} = 0 = \overline{a}b \tag{1}$$

for all $a, b \in A$. The main result of this section is:

Theorem 1 *Let $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ be an oriented quantum algebra over the field k, let $\mathcal{A} = A \oplus A^{op}$ be the direct product of A and A^{op} and write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$, $\rho^{-1} = \sum_{\jmath=1}^{s} \alpha_\jmath \otimes \beta_\jmath$. Then:*

a) *$(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s})$ is a quantum algebra over k, where*

$$\boldsymbol{\rho} = \sum_{\imath=1}^{r} (a_\imath \otimes b_\imath + \overline{a_\imath} \otimes \overline{b_\imath}) + \sum_{\jmath=1}^{s} (\overline{\alpha_\jmath} \otimes \beta_\jmath + \alpha_\jmath \otimes \overline{t_{\mathsf{d}}^{-1} \circ t_{\mathsf{u}}^{-1}(\beta_\jmath)})$$

and $\boldsymbol{s}(a \oplus b) = b \oplus t_{\mathsf{d}}^{-1} \circ t_{\mathsf{u}}^{-1}(a)$ for all $a, b \in A$.

b) *$(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{t}_{\mathsf{d}}, \boldsymbol{t}_{\mathsf{u}})$ is an oriented quantum algebra over k, $\boldsymbol{t}_{\mathsf{d}}, \boldsymbol{t}_{\mathsf{u}}$ commute with $\boldsymbol{s}$ and $\boldsymbol{t}_{\mathsf{d}} \circ \boldsymbol{t}_{\mathsf{u}} = \boldsymbol{s}^{-2}$, where $\boldsymbol{t}_{\mathsf{d}}(a \oplus b) = t_{\mathsf{d}}(a) \oplus t_{\mathsf{d}}(b)$ and $\boldsymbol{t}_{\mathsf{u}}(a \oplus b) = t_{\mathsf{u}}(a) \oplus t_{\mathsf{u}}(b)$ for all $a, b \in A$.*

c) *The projection $\pi : \mathcal{A} \longrightarrow A$ onto the first factor determines a morphism $\pi : (\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{t}_{\mathsf{d}}, \boldsymbol{t}_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ of oriented quantum algebras.*

PROOF: This result was announced as [16, Theorem 2] and the proof here was also given in that paper. We repeat the proof here for the reader's convenience and to connect it to a proof of Theorem 3.

Part b) is a straightforward calculation which is left to the reader and part c) follows by definitions. As for part a) we may assume that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) = (A, \rho, 1_A, t)$ is standard. In this case

$$\boldsymbol{\rho} = \sum_{\imath=1}^{r} (a_\imath \otimes b_\imath + \overline{a_\imath} \otimes \overline{b_\imath}) + \sum_{\jmath=1}^{s} (\overline{\alpha_\jmath} \otimes \beta_\jmath + \alpha_\jmath \otimes \overline{t^{-1}(\beta_\jmath)}) \quad \text{and} \quad \boldsymbol{s}(a \oplus b) = b \oplus t^{-1}(a)$$

for all $a, b \in A$. Since t is an algebra automorphism of A it follows that t^{-1} is also. Thus $\boldsymbol{s} : \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is an algebra isomorphism. By definition $\boldsymbol{s}(a) = \overline{t^{-1}(a)}$ and $\boldsymbol{s}(\overline{a}) = a$ for all $a \in A$. Since $\rho = (t \otimes t)(\rho)$ it follows that $\rho^{-1} = (t^{-1} \otimes t^{-1})(\rho^{-1})$. At this point it is easy to see that $\boldsymbol{\rho} = (\boldsymbol{s} \otimes \boldsymbol{s})(\boldsymbol{\rho})$, or

(QA.2) is satisfied for $\boldsymbol{\rho}$ and $\boldsymbol{s}$. Using the equation $\rho^{-1} = (t^{-1}\otimes t^{-1})(\rho^{-1})$ we calculate

$$(s\otimes 1_{\mathcal{A}})(\boldsymbol{\rho}) = \sum_{\imath=1}^{r}\left(\overline{t^{-1}(a_\imath)}\otimes b_\imath + a_\imath\otimes\overline{b_\imath}\right) + \sum_{\jmath=1}^{s}\left(\alpha_\jmath\otimes\beta_\jmath + \overline{\alpha_\jmath}\otimes\overline{\beta_\jmath}\right).$$

Using (1), the equation $(t^{-1}\otimes 1_A)(\rho) = (1_A\otimes t)(\rho)$, which follows by (qa.2), we see that

$$\boldsymbol{\rho}((s\otimes 1_{\mathcal{A}})(\boldsymbol{\rho})) = 1\otimes 1 + \overline{1}\otimes\overline{1} + \overline{1}\otimes 1 + 1\otimes\overline{1} = 1_{\mathcal{A}}\otimes 1_{\mathcal{A}} = ((s\otimes 1_{\mathcal{A}})(\boldsymbol{\rho}))\boldsymbol{\rho}.$$

Therefore $\boldsymbol{\rho}$ is invertible and $\boldsymbol{\rho}^{-1} = (\boldsymbol{s}\otimes 1_A)(\boldsymbol{\rho})$. We have shown that (QA.1) holds for $\boldsymbol{\rho}$ and $\boldsymbol{s}$.

That $\boldsymbol{\rho}$ satisfies (QA.3) is a rather lengthy and interesting calculation. It is a straightforward exercise to see that (QA.3) for $\boldsymbol{\rho}$ is equivalent to a set of eight equations. With the notation convention $(\rho^{-1})_{\imath\jmath} = \rho^{-1}_{\imath\jmath}$ for $1 \leq \imath < \jmath \leq 3$, this set of eight equations can be rewritten as set of six equations which are:

$$\rho_{1\,2}\rho_{1\,3}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,3}\rho_{1\,2}, \tag{2}$$

$$\rho_{1\,2}\rho^{-1}_{2\,3}\rho^{-1}_{1\,3} = \rho^{-1}_{1\,3}\rho^{-1}_{2\,3}\rho_{1\,2}, \tag{3}$$

$$\rho^{-1}_{1\,3}\rho^{-1}_{1\,2}\rho_{2\,3} = \rho_{2\,3}\rho^{-1}_{1\,2}\rho^{-1}_{1\,3}, \tag{4}$$

$$\sum_{\ell=1}^{r}\sum_{\jmath,m=1}^{s} a_\ell\alpha_\jmath\otimes\beta_\jmath\alpha_m\otimes t^{-1}(\beta_m)b_\ell = \sum_{\jmath,m=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m), \tag{5}$$

$$\sum_{\jmath,m=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath a_\ell\otimes\alpha_m t^{-1}(\beta_\jmath)\otimes b_\ell\beta_m = \sum_{\ell=1}^{r}\sum_{\jmath,m=1}^{s} a_\ell\alpha_\jmath\otimes t^{-1}(\beta_\jmath)\alpha_m\otimes\beta_m b_\ell \tag{6}$$

and

$$\sum_{\jmath,\ell=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath\alpha_\ell\otimes a_m t^{-1}(\beta_\jmath)\otimes b_m t^{-1}(\beta_\ell) = \sum_{\ell,\jmath=1}^{s}\sum_{m=1}^{r}\alpha_\ell\alpha_\jmath\otimes t^{-1}(\beta_\jmath)a_m\otimes t^{-1}(\beta_\ell)b_m. \tag{7}$$

By assumption (2) holds. Since $\rho_{\imath\jmath}$ is invertible and $(\rho_{\imath\jmath})^{-1} = (\rho^{-1})_{\imath\jmath} = \rho^{-1}_{\imath\jmath}$, equations (3)–(4) hold since (2) does.

We note that t^{-1} is an algebra automorphism of A and $\rho^{-1} = (t^{-1}\otimes t^{-1})(\rho^{-1})$. Thus applying $1_A\otimes t^{-1}\otimes 1_A$ to both sides of the equation of (5) we see that (5) and (6) are equivalent; applying $t^{-1}\otimes 1_A\otimes 1$ to both sides of (7) we see

that (7) is equivalent to $\rho_{2\,3}\rho_{1\,2}^{-1}\rho_{1\,3}^{-1} = \rho_{1\,3}^{-1}\rho_{1\,2}^{-1}\rho_{2\,3}$, a consequence of (2). To complete the proof of part a) we need only show that (5) holds.

By assumption $(1_A \otimes t)(\rho)$ and ρ^{-1} are inverses in $A\otimes A^{op}$. Thus ρ and $(1_A\otimes t^{-1})(\rho)$ are inverses in $A\otimes A^{op}$ as $1_A\otimes t^{-1}$ is an algebra endomorphism of $A\otimes A^{op}$. Recall that ρ^{-1} satisfies (QA.3). Therefore

$$
\begin{aligned}
&\sum_{\jmath,m=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m)\\
&\quad= \sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}(a_v\alpha_u)\alpha_\jmath a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m)(t^{-1}(\beta_u)b_v)\\
&\quad= \sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}a_v(\alpha_u\alpha_\jmath)a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m\beta_u)b_v\\
&\quad= \sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}a_v(\alpha_\jmath\alpha_u)a_\ell\otimes\beta_\jmath\alpha_m\otimes b_\ell t^{-1}(\beta_u\beta_m)b_v\\
&\quad= \sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}a_v\alpha_\jmath(\alpha_u a_\ell)\otimes\beta_\jmath\alpha_m\otimes(b_\ell t^{-1}(\beta_u))t^{-1}(\beta_m)b_v\\
&\quad= \sum_{v=1}^{r}\sum_{\jmath,m=1}^{s}a_v\alpha_\jmath\otimes\beta_\jmath\alpha_m\otimes t^{-1}(\beta_m)b_v.
\end{aligned}
$$

which establishes (5). $\square$

Denote by $\mathcal{C}_q$ the category whose objects are quintuples $(A,\rho,s,t_{\mathsf{d}},t_{\mathsf{u}})$, where (A,ρ,s) is a quantum algebra over k and $(A,\rho,t_{\mathsf{d}},t_{\mathsf{u}})$ is an oriented quantum algebra over k such that $t_{\mathsf{d}},t_{\mathsf{u}}$ commute with s and $t_{\mathsf{d}}\circ t_{\mathsf{u}} = s^{-2}$, and whose morphisms $f : (A,\rho,s,t_{\mathsf{d}},t_{\mathsf{u}}) \longrightarrow (A',\rho',s',t'_{\mathsf{d}},t'_{\mathsf{u}})$ are algebra maps $f : A \longrightarrow A'$ which determine morphisms $f : (A,\rho,s) \longrightarrow (A',\rho',s')$ and $f : (A,\rho,t_{\mathsf{d}},t_{\mathsf{u}}) \longrightarrow (A',\rho',t'_{\mathsf{d}},t'_{\mathsf{u}})$. The construction $(\mathcal{A},\boldsymbol{\rho},\boldsymbol{s},\boldsymbol{t}_{\mathsf{d}},\boldsymbol{t}_{\mathsf{d}})$ of Theorem 1 is a cofree object of $\mathcal{C}_q$. Let $\pi : \mathcal{A} \longrightarrow A$ be the projection onto the first factor.

Proposition 3 *Let $(A,\rho,t_{\mathsf{d}},t_{\mathsf{u}})$ be an oriented quantum algebra over the field k. Then the pair $((\mathcal{A},\boldsymbol{\rho},\boldsymbol{s},\boldsymbol{t}_{\mathsf{d}},\boldsymbol{t}_{\mathsf{u}}),\pi)$ satisfies the following properties:*

a) $(\mathcal{A},\boldsymbol{\rho},\boldsymbol{s},\boldsymbol{t}_{\mathsf{d}},\boldsymbol{t}_{\mathsf{u}})$ *is an object of* $\mathcal{C}_q$ *and* $\pi : (\mathcal{A},\boldsymbol{\rho},\boldsymbol{t}_{\mathsf{d}},\boldsymbol{t}_{\mathsf{u}}) \longrightarrow (A,\rho,t_{\mathsf{d}},t_{\mathsf{u}})$ *is a morphism of oriented quantum algebras.*

b) *Suppose that* $(A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ *is an object of* $\mathcal{C}_q$ *and suppose that* $f : (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *is a morphism of oriented quantum algebras. Then there is a morphism* $F : (A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (\boldsymbol{\mathcal{A}}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{t}_{\mathsf{d}}, \boldsymbol{t}_{\mathsf{u}})$ *uniquely determined by* $\pi \circ F = f$.

PROOF: We have shown part a). Let $f : (A', \rho, t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ be a morphism of oriented quantum algebras. To show part b) we first suppose that $F : (A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (\boldsymbol{\mathcal{A}}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{t}_{\mathsf{d}}, \boldsymbol{t}_{\mathsf{u}})$ is a morphism which satisfies $\pi \circ F = f$. Now there are linear maps $g, h : A' \longrightarrow A$ such that $F(x) = g(x) \oplus h(x)$ for all $x \in A'$. Since $\pi \circ F = f$ it follows that $g = f$. Since $s \circ F = F \circ s'$ it follows that $h = g \circ s' = f \circ s'$. Thus $F(x) = f(x) \oplus f(s'(x))$ for all $x \in A'$ which establishes the uniqueness assertion of part b).

To establish the existence assertion of part b), we consider the algebra homomorphism $F : A' \longrightarrow A$ defined by $F(x) = f(x) \oplus f(s'(x))$ for all $x \in A'$. Since $f : (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is a morphism $f \circ t'_{\mathsf{d}} = t_{\mathsf{d}} \circ f$ and $f \circ t'_{\mathsf{u}} = t_{\mathsf{u}} \circ f$. Therefore

$$
\begin{aligned}
\boldsymbol{s} \circ F(x) &= f(s'(x)) \oplus (t_{\mathsf{d}}^{-1} \circ t_{\mathsf{u}}^{-1} \circ f)(x) \\
&= f(s'(x)) \oplus (f \circ t_{\mathsf{d}}'^{-1} \circ t_{\mathsf{u}}'^{-1})(x) \\
&= f(s'(x)) \oplus f((s'^2(x)) \\
&= F \circ s'(x)
\end{aligned}
$$

for all $x \in A'$ which shows that $\boldsymbol{s} \circ F = F \circ s'$. Since t'_{d} and s' commute the calculation

$$
\begin{aligned}
\boldsymbol{t}_d \circ F(x) &= t_{\mathsf{d}}(f(x)) \oplus t_{\mathsf{d}}(f(s'(x))) \\
&= f(t'_{\mathsf{d}}(x)) \oplus f(t'_{\mathsf{d}}(s'(x))) \\
&= f(t'_{\mathsf{d}}(x)) \oplus f(s'(t'_{\mathsf{d}}(x))) \\
&= F \circ t'_{\mathsf{d}}(x)
\end{aligned}
$$

for all $x \in A'$ establishes $\boldsymbol{t}_d \circ F = F \circ t'_{\mathsf{d}}$. Likewise $\boldsymbol{t}_u \circ F = F \circ t'_{\mathsf{u}}$.

To complete the proof that $F : (A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (\boldsymbol{\mathcal{A}}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{T}_d, \boldsymbol{T}_d)$ is a morphism, and thus to complete the proof of the proposition, we need only show that $\boldsymbol{\rho} = (F \oplus F)(\boldsymbol{\rho})$. Since $f : (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is a morphism of oriented quantum algebras $\rho = (f \otimes f)(\rho')$ and thus $\rho^{-1} = (f \oplus f)(\rho'^{-1})$. Now $f \circ s'^2 = t_{\mathsf{d}}^{-1} \circ t_{\mathsf{u}}^{-1} \circ f$ follows from the hypothesis of part b).

Write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$ and $\rho' = \sum_{\imath'=1}^{r'} a'_{\imath'} \otimes b_{\imath'}$. Using the fact that (A', ρ', s') is a quantum algebra we can now calculate

$$
\begin{aligned}
(F\otimes F)(\rho') &= \sum_{\imath'=1}^{r'} F(a'_{\imath'})\otimes F(b'_{\imath'}) \\
&= \sum_{\imath'=1}^{r'} (f(a'_{\imath'})\oplus f(s'(a'_{\imath'})))\otimes(f(b'_{\imath'})\oplus f(s'(b'_{\imath'}))) \\
&= \sum_{\imath'=1}^{r'} \left(f(a'_{\imath'}) + \overline{f(s'(a'_{\imath'}))}\right)\otimes\left(f(b'_{\imath'}) + \overline{f(s'(b'_{\imath'}))}\right) \\
&= \sum_{\imath'=1}^{r'} \left(f(a'_{\imath'})\otimes f(b'_{\imath'}) + \overline{f(s'(a'_{\imath'}))}\otimes\overline{f(s'(b'_{\imath'}))}\right) \\
&\qquad + \sum_{\imath'=1}^{r'} \left(\overline{f(s(a'_{\imath'}))}\otimes f(b'_{\imath'}) + f(a'_{\imath'})\otimes\overline{f(s'(b'_{\imath'}))}\right) \\
&= \sum_{\imath'=1}^{r'} \left(f(a'_{\imath'})\otimes f(b'_{\imath'}) + \overline{f(a'_{\imath'})}\otimes\overline{f(b'_{\imath'})}\right) \\
&\qquad + \sum_{\imath'=1}^{r'} \left(\overline{f(s(a'_{\imath'}))}\otimes f(b'_{\imath'}) + f(s'(a'_{\imath'}))\otimes\overline{f(s'^2(b'_{\imath'}))}\right) \\
&= \sum_{\imath'=1}^{r'} \left(f(a'_{\imath'})\otimes f(b'_{\imath'}) + \overline{f(a'_{\imath'})}\otimes\overline{f(b'_{\imath'})}\right) \\
&\qquad + \sum_{\imath'=1}^{r'} \left(\overline{f(s(a'_{\imath'}))}\otimes f(b'_{\imath'}) + f(s'(a'_{\imath'}))\otimes\overline{t_{\mathsf{d}}^{-1}\circ t_{\mathsf{u}}^{-1}(f(b'_{\imath'}))}\right) \\
&= \sum_{\imath=1}^{r} \left(a_\imath\otimes b_\imath + \overline{a_\imath}\otimes\overline{b_\imath}\right) + \sum_{\jmath=1}^{s}\left(\overline{\alpha_\jmath}\otimes\beta_\jmath + \alpha_\jmath\otimes\overline{t_{\mathsf{d}}^{-1}\circ t_{\mathsf{u}}^{-1}(\beta_\jmath)}\right) \\
&= \boldsymbol{\rho}.
\end{aligned}
$$

□

4 General Results for Oriented Quantum Coalgebras

In this section we develop some of the basic theory of oriented quantum coalgebras. Our discussion parallels that of [11, Sections 4 and 5] to a good extent. Proofs of most assertions made in this section can be obtained by modifying the proofs of corresponding statements in [11] about quantum coalgebras. Thus we shall tend to omit many details here.

Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k and let $b', b'' : C \times C^{cop} \longrightarrow k$ be the bilinear forms defined by $b'(c, d) = b(c, T_{\mathsf{u}}(d))$ and $b''(c, d) = b^{-1}(T_{\mathsf{d}}(c), d)$ for all $c, d \in C$. Then the two equations of (qc.1) may be viewed as technical formulations of the statements b'' is a left inverse for b' and b'' is a right inverse for b'. When C is finite-dimensional the two equations of (qc.1) are equivalent. Thus in the finite-dimensional case, axiom (qc.1) for oriented quantum coalgebra can be simplified.

A straightforward calculation shows that $(C^{cop}, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is an oriented quantum coalgebra over k. Let $T = T_{\mathsf{d}}$ or $T = T_{\mathsf{u}}$. Since T^{-1} is a coalgebra automorphism of C with respect to $\{b, b^{-1}\}$, by part a) of Lemma 1 and the equation $b^{-1}(T(c), T(d)) = b^{-1}(c, d)$ for all $c, d \in C$, it follows that $(C, b^{-1}, T_{\mathsf{d}}^{-1}, T_{\mathsf{u}}^{-1})$ is an oriented quantum coalgebra over k. See the proof of the corresponding statement for quantum coalgebras given in [11, Section 4.2]. Let $b^{op} : C \times C \longrightarrow k$ be the bilinear form defined by $b^{op}(c, d) = b(d, c)$ for all $c, d \in C$. Then $(C, b^{op}, T_{\mathsf{u}}, T_{\mathsf{d}})$ is an oriented quantum coalgebra over k as well.

Let K be a field extension of k. Then $(C \otimes K, b \otimes 1_{K \otimes K}, T_d \otimes 1_K, T_{\mathsf{u}} \otimes 1_K)$ is an oriented quantum coalgebra over K, where $b \otimes 1_{K \otimes K}(c \otimes \alpha, d \otimes \beta) = \alpha\beta b(c, d)$ for all $c, d \in C$ and $\alpha, \beta \in K$. Suppose that $(C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ is also an oriented quantum coalgebra over k. Then $(C \otimes C', b'', T_{\mathsf{d}} \otimes T'_{\mathsf{d}}, T_{\mathsf{u}} \otimes T'_{\mathsf{u}})$ is an oriented quantum coalgebra over k, called the *tensor product of oriented quantum coalgebras over* k, where $b''(c \otimes c', d \otimes d') = b(c, d)b'(c', d')$ for all $c, d \in C$ and $c', d' \in C'$.

An *oriented quantum subcoalgebra of* $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is an oriented quantum coalgebra $(D, b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$, where D is a subcoalgebra of C and the inclusion $\imath : D \longrightarrow C$ determines a morphism $\imath : (D, b', T'_{\mathsf{d}}, T'_{\mathsf{u}}) \longrightarrow (C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$. In this case $b' = b|_{D \times D}$, $T_{\mathsf{d}}(D) = D = T_{\mathsf{u}}(D)$ and $T'_{\mathsf{d}} = T_{\mathsf{d}}|_D$, $T'_{\mathsf{u}} = T_{\mathsf{u}}|_D$. Conversely, if D is a subcoalgebra of C and $T_{\mathsf{d}}(D) = D = T_{\mathsf{u}}(D)$, then

$(D, b_{D\times D}, T_{\mathsf{d}}|_D, T_{\mathsf{u}}|_D)$ is an oriented quantum subcoalgebra of $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$.

Let I be a coideal of C which satisfies $T_{\mathsf{d}}(I) = I = T_{\mathsf{u}}(I)$ and $b(I, C) = (0) = b(C, I)$. Then the quotient C/I has a unique oriented quantum coalgebra structure $(C/I, \overline{b}, \overline{T}_{\mathsf{d}}, \overline{T}_{\mathsf{u}})$, which we refer to as a *quotient oriented quantum coalgebra structure*, such that the projection $\pi : C \longrightarrow C/I$ induces a morphism $\pi : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C/I, \overline{b}, \overline{T}_{\mathsf{d}}, \overline{T}_{\mathsf{u}})$.

Let I be the sum of the coideals J of C such that $T_{\mathsf{d}}(J) = J = T_{\mathsf{u}}(J)$ and $b(J, C) = (0) = b(C, J)$. Then I is a coideal of C which satisfies $T_{\mathsf{d}}(I) = I = T_{\mathsf{u}}(I)$ and $b(I, C) = (0) = b(C, I)$. Set $C_r = C/I$. The quotient oriented quantum coalgebras structure $(C_r, b_r, T_{r\,\mathsf{d}}, T_{r\,\mathsf{u}})$ is the coalgebra counterpart of the minimal oriented quantum subalgebra $(A_\rho, \rho, t_{\mathsf{d}}|_{A_\rho}, t_{\mathsf{u}}|_{A_\rho})$ of an oriented quantum algebra $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ over k. Observe that if J is a coideal of C_r such that $T_{r\,\mathsf{d}}(J) = J = T_{r\,\mathsf{u}}(J)$ and $b_r(J, C_r) = (0) = b_r(C_r, J)$ then $J = (0)$.

Recall that a bilinear form $\beta : V\times V \longrightarrow k$ define for a vector space over k is *left* (respectively *right*) *non-singular* if $\beta_{(\ell)}$ (respectively $\beta_{(r)}$) is one-one. If b is either left non-singular or right non-singular then $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is strict.

Lemma 2 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k and suppose that b is either left non-singular or is right non-singular. Then T_{d} and T_{u} are coalgebra automorphisms of C.*

PROOF: Consider the linear map $b_{(\ell)}\otimes b_{(\ell)} : C\otimes C \longrightarrow C^*\otimes C^*$ and regard $C^*\otimes C^*$ as a subspace of $(C\otimes C)^*$ by $(c^*\otimes d^*)(c\otimes d) = c^*(c)d^*(d)$ for all $c^*, d^* \in C^*$ and $c, d \in C$. Let $T = T_{\mathsf{d}}$ or $T = T_{\mathsf{u}}$. Then

$$b(d, T(c_{(1)}))b(e, T(c_{(2)})) = b(d, T(c)_{(1)})b(e, T(c)_{(2)})$$

for all $d, c, e \in C$ which holds if and only if

$$b_{(\ell)}\otimes b_{(\ell)}(T(c_{(1)})\otimes T(c_{(2)})) = b_{(\ell)}\otimes b_{(\ell)}(T(c)_{(1)}\otimes T(c)_{(2)})$$

for all $c \in C$. Since $\epsilon\circ T = \epsilon$, it follows that T is a coalgebra automorphism of C if $b_{(\ell)}$ is one-one. Likewise T is a coalgebra automorphism of C if $b_{(r)}$ is one-one. $\Box$

Oriented quantum coalgebra structures can be pulled back just as quantum coalgebra structures can be pulled back.

Theorem 2 *Suppose that $\pi : C \longrightarrow C'$ is an onto map of coalgebras over the field k and suppose that $(C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ is an oriented quantum coalgebra structure on C. Then there exists an oriented quantum coalgebra structure $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ on C such that $\pi : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ is a morphism.*

PROOF: The proof boils down to finding commuting linear automorphisms $T_{\mathsf{d}}, T_{\mathsf{u}}$ of C which satisfies $T'_{\mathsf{d}} \circ \pi = \pi \circ T_{\mathsf{d}}$ and $T'_{\mathsf{u}} \circ \pi = \pi \circ T_{\mathsf{u}}$. This is easy enough to do. The reader may want to refer to the proof of the corresponding result for quantum coalgebras [11, Theorem 2]. $\square$

By [12, Proposition 1.1.1], for example, every finite-dimensional coalgebra over k is the quotient of $\mathrm{C}_n(k)$ for some $n \geq 1$. Thus as a corollary to Theorem 2:

Corollary 3 *Every finite-dimensional oriented quantum coalgebra over the field k is the quotient of an oriented quantum coalgebra structure on $\mathrm{C}_n(k)$ for some $n \geq 1$.*

$\square$

There is an analog of [8, Proposition 2] for quantum coalgebras.

Proposition 4 *If (C, b, S) is a quantum (respectively strict quantum) coalgebra over k then $(C, b, 1_C, S^{-2})$ is an oriented (respectively strict oriented) quantum coalgebra over k.*

PROOF: Let (C, b, S) be a quantum coalgebra over k. To prove the proposition we need only show that $(C, b, 1_C, S^{-2})$ is an oriented quantum coalgebra over k. Since $S : C \longrightarrow C^{cop}$ is a coalgebra isomorphism with respect to b and (QC.1), (QC.2) hold for S it follows that S^{-2} is a coalgebra automorphism of C with respect to $\{b, b^{-1}\}$. See the proof of Lemma 1. Since (QC.2) holds for S it follows that (qc.2) holds for S^{-2}. Now (QC.3) is (qc.3). Thus to complete the proof we need only show that (qc.1) holds for $T_{\mathsf{d}} = 1_C$ and $T_{\mathsf{d}} = S^{-2}$. The calculation

$$\begin{aligned} b(c_{(1)}, S^{-2}(d_{(2)}))b^{-1}(c_{(2)}, d_{(1)}) &= b(S(c_{(1)}), S^{-1}(d_{(2)}))b(S(c_{(2)}), d_{(1)}) \\ &= b(S(c)_{(2)}, S^{-1}(d_{(2)}))b(S(c)_{(1)}, d_{(1)}) \\ &= b^{-1}(S(c)_{(2)}, d_{(2)})b(S(c)_{(1)}, d_{(1)}) \\ &= \epsilon(S(c))\epsilon(d) \\ &= \epsilon(c)\epsilon(d) \end{aligned}$$

for all $c, d \in C$ shows that $b(c_{(1)}, S^{-2}(d_{(2)}))b^{-1}(c_{(2)}, d_{(1)}) = \epsilon(c)\epsilon(d)$ for all $c, d \in C$. Likewise $b^{-1}(c_{(1)}, d_{(2)})b(c_{(2)}, S^{-2}(d_{(1)})) = \epsilon(c)\epsilon(d)$ for all $c, d \in C$. We have established (qc.1) for $T_{\mathsf{d}} = 1_C$ and $T_{\mathsf{u}} = S^{-2}$. $\Box$

By virtue of the preceding proposition every quantum coalgebra over k has the structure of a standard oriented quantum coalgebra. Every oriented quantum coalgebra over k does also by the analog of [8, Proposition 1].

Proposition 5 *If $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is an oriented quantum coalgebra over k then $(C, b, T_{\mathsf{d}}\circ T_{\mathsf{u}}, 1_C)$ and $(C, b, 1_C, T_{\mathsf{d}}\circ T_{\mathsf{u}})$ are also.*

PROOF: Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over k. By Lemma 1 the composition $T_{\mathsf{d}}\circ T_{\mathsf{u}}$ is a coalgebra automorphism of C with respect to $\{b, b^{-1}\}$. The proof boils down to showing that (qc.1) holds for $(C, b, T_{\mathsf{d}}\circ T_{\mathsf{u}}, 1_C)$ and $(C, b, 1_C, T_{\mathsf{d}}\circ T_{\mathsf{u}})$. To show that (qc.1) holds for $(C, b, T_{\mathsf{d}}\circ T_{\mathsf{u}}, 1_C)$ we note that

$$\begin{aligned}
&b(c_{(1)}, d_{(2)})b^{-1}(T_{\mathsf{d}}\circ T_{\mathsf{u}}(c_{(2)}), d_{(1)}) \\
&\quad = \; b(T_{\mathsf{u}}(c_{(1)}), T_{\mathsf{u}}(d_{(2)}))b^{-1}(T_{\mathsf{u}}(c_{(2)}), T_{\mathsf{d}}^{-1}(d_{(1)})) \\
&\quad = \; b(T_{\mathsf{u}}(c)_{(1)}, T_{\mathsf{u}}(d_{(2)}))b^{-1}(T_{\mathsf{u}}(c)_{(2)}, T_{\mathsf{d}}^{-1}(d_{(1)})) \\
&\quad = \; b(T_{\mathsf{u}}(c)_{(1)}, T_{\mathsf{u}}(d_{(2)}))b^{-1}(T_{\mathsf{d}}(T_{\mathsf{u}}(c)_{(2)}), d_{(1)}) \\
&\quad = \; \epsilon(T_{\mathsf{u}}(c))\epsilon(d) \\
&\quad = \; \epsilon(c)\epsilon(d)
\end{aligned}$$

for all $c, d \in C$ and likewise $b^{-1}(T_{\mathsf{d}}\circ T_{\mathsf{u}}(c_{(1)}), d_{(2)})b(c_{(2)}, d_{(1)}) = \epsilon(c)\epsilon(d)$ for all $c, d \in C$. Similar calculations show that (qc.1) holds for $(C, b, 1_C, T_{\mathsf{d}}\circ T_{\mathsf{u}})$ also. The fact that T_{d} and T_{u} commute is used in the latter. $\Box$

5 A Basic Relationship Between Oriented and Unoriented Quantum Coalgebra Structures

Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k and let $\mathcal{C} = C \oplus C^{cop}$ be the direct sum of the coalgebras C and C^{cop}. There is a quantum coalgebra structure $(\mathcal{C}, \boldsymbol{\beta}, \mathbf{S})$ on $\mathcal{C}$ which is accounted for by [5, Theorem 1] and there is a coalgebra counterpart of Theorem 1.

Let $\imath : C \longrightarrow \mathcal{C}$ be the one-one map defined by $\imath(c) = c\oplus 0$ for all $c \in C$, make the identification $c = \imath(c)$ for all $c \in C$ and define $\overline{c\oplus d} = d\oplus c$ for all $c, d \in C$.

Theorem 3 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k and let $\mathcal{C} = C\oplus C^{cop}$ be the direct sum of C and C^{cop}. Then:*

a) $(\mathcal{C}, \boldsymbol{\beta}, \mathbf{S})$ *is a quantum coalgebra over k, where $\boldsymbol{\beta}$ is determined by*

$$\boldsymbol{\beta}(c, d) = b(c, d) = \boldsymbol{\beta}(\overline{c}, \overline{d}), \quad \boldsymbol{\beta}(\overline{c}, d) = b^{-1}(c, d),$$

$$\boldsymbol{\beta}(c, \overline{d}) = b^{-1}(c, T^{-2}(d)) \;\; \textit{and} \;\; \mathbf{S}(c\oplus d) = T_{\mathsf{d}}^{-1}\circ T_{\mathsf{u}}^{-1}(d)\oplus c$$

for all $c, d \in C$.

b) $(\mathcal{C}, \boldsymbol{\beta}, \mathbf{T}_{\mathsf{d}}, \mathbf{T}_{\mathsf{u}})$ *is an oriented quantum coalgebra over k and $\mathbf{T}_{\mathsf{d}}, \mathbf{T}_{\mathsf{u}}$ commute with $\mathbf{S}$, where $\mathbf{T}_{\mathsf{d}}(c\oplus d) = T_{\mathsf{d}}(c)\oplus T_{\mathsf{d}}(d)$ and $\mathbf{T}_{\mathsf{u}}(c\oplus d) = T_{\mathsf{u}}(c)\oplus T_{\mathsf{u}}(d)$ for all $c, d \in C$.*

c) *The inclusion $\imath : C \longrightarrow \mathcal{C}$ induces a morphism of oriented quantum coalgebras $\imath : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (\mathcal{C}, \boldsymbol{\beta}, \mathbf{T}_{\mathsf{d}}, \mathbf{T}_{\mathsf{u}})$.*

PROOF: The proofs of parts b) and c) are straightforward and are left to the reader. As for part a), we first note that $(C, b, 1_C, T_{\mathsf{d}}\circ T_{\mathsf{u}})$ is an oriented quantum coalgebra over k by Proposition 5 and that $(C, b, (T_{\mathsf{d}}\circ T_{\mathsf{u}})^{-1})$ is a $(T_{\mathsf{d}}\circ T_{\mathsf{u}})^{-1}$-form structure [5, Section 3]. Thus part a) follows by [5, Theorem 1]. $\square$

The proof of [5, Theorem 1] is conceptually far more difficult than the proof of Theorem 1. The formulation of [5, Theorem 1] preceded the definitions of oriented quantum algebra and oriented quantum coalgebra. The motivation for this theorem was to simplify calculation of invariants of 1–1 tangles which arise from certain quantum coalgebras.

Let $\mathcal{C}_{cq}$ be the category whose objects are quintuples $(C, b, S, T_{\mathsf{d}}, T_{\mathsf{u}})$, where (C, b, S) is a quantum coalgebra over k, $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is an oriented quantum coalgebra over k and T_{d}, T_{u} commute with S, and whose morphisms $f : (C, b, S, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', S', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ are morphisms of quantum coalgebras $f : (C, b, S) \longrightarrow (C', b', S')$ and morphisms of oriented quantum coalgebras $f : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$. Our construction gives rise to a free object of $\mathcal{C}_{cq}$. The following result, whose proof is left to the reader, is a coalgebra counterpart of Proposition 3.

Proposition 6 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k. Then the pair $(\imath, (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}}))$ satisfies the following properties:*

a) $(\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}})$ *is an object of* $\mathcal{C}_{cq}$ *and* $\imath : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}})$ *is a morphism of oriented quantum coalgebras over k.*

b) *Suppose that* $(C', b', S', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *is an object of* $\mathcal{C}_{cq}$ *and* $f : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *is a morphism of oriented quantum coalgebras over k. There is a morphism* $F : (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}}) \longrightarrow (C', b', S', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *uniquely determined by* $F \circ \imath = f$.

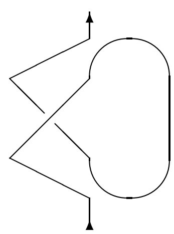
□

6 A Regular Isotopy Invariant of Oriented 1–1 Tangles Which Arises from an Oriented Quantum Coalgebra

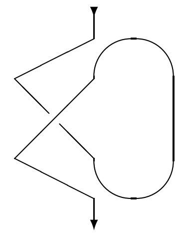
In this section we construct a regular isotopy invariant $\mathbf{Inv}_C$ of oriented 1–1 tangle diagrams from an oriented quantum coalgebra C over k in much the same manner that we constructed an invariant of 1–1 tangle diagrams from a quantum coalgebra over k in [11, Section 6.1]. The invariant we construct can be considered the coalgebra version of the invariant of oriented 1–1 tangle diagrams described in [9, Section 1] and [8, Section 3] which arises from an oriented quantum algebra. In Section 6.1 we describe $\mathbf{Inv}_C$ and in Section 6.2 we prove that $\mathbf{Inv}_C$ is a regular isotopy invariant of oriented 1–1 tangle diagrams (and thus determines a regular isotopy invariant of oriented 1–1 tangles).

6.1 Invariants of Oriented 1–1 Tangle Diagrams Arising from Oriented Quantum Coalgebras

We represent oriented 1–1 tangles as diagrams in the plane with respect to the vertical direction. Simple examples of these diagrams are

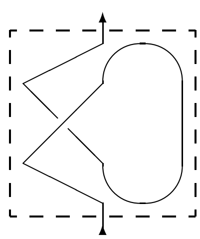


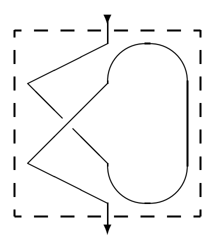
and



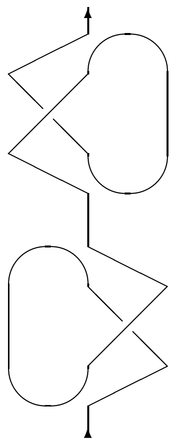
where the arrow heads indicate orientation. We shall refer to the tangle diagrams above as $\mathbf{T}_{\text{curl}}$ and $\mathbf{T}_{\text{curl}}^{op}$ respectively. We let **Tang** denote the set of all oriented 1–1 tangle diagrams. If $\mathbf{T} \in \mathbf{Tang}$ then $\mathbf{T}^{op}$ is the underlying diagram of $\mathbf{T}$ with the opposite orientation.

The point on the tangle diagram at which a traversal of the diagram in the direction of the orientation begins is called the *base point of the diagram* and the point at which such a traversal ends is called the *end point of the diagram*. We require 1–1 tangle diagrams to be completely contained in a box except for two protruding line segments as indicated by the two examples below.

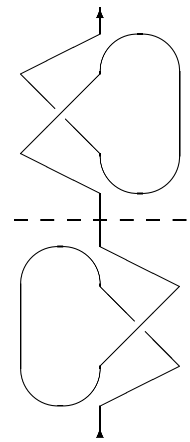




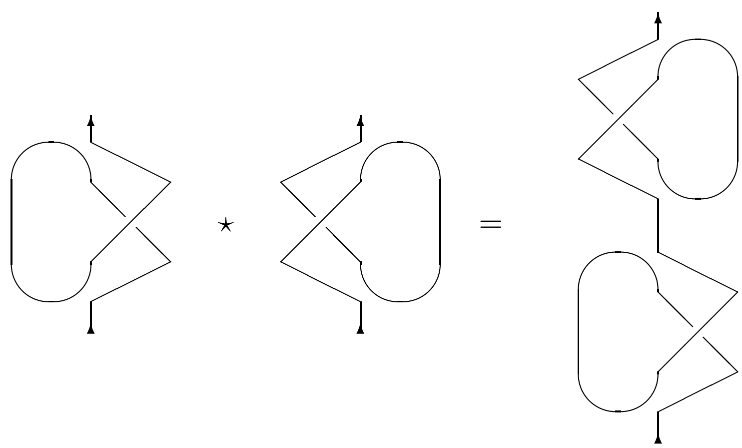
When an oriented 1–1 tangle diagram $\mathbf{T}$ can be written as the union of two 1–1 tangle diagrams $\mathbf{T}_1$ and $\mathbf{T}_2$, where the end point of $\mathbf{T}_1$ is the base point of $\mathbf{T}_2$, and the horizontal line passing through this common point otherwise separates $\mathbf{T}_1$ and $\mathbf{T}_2$, then $\mathbf{T}$ is called the *product of* $\mathbf{T}_1$ and $\mathbf{T}_2$ and we write $\mathbf{T} = \mathbf{T}_1 \star \mathbf{T}_2$. For example,



may be split into two parts



and thus



Multiplication is clearly an associative operation.

Oriented 1–1 tangle diagrams consist of some or all of the following components:

- oriented crossings;

    under crossings

over crossings

- oriented local extrema;

local maxima 

local minima 

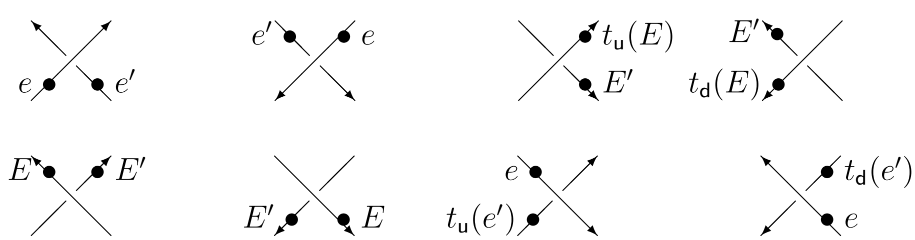
and

- oriented "vertical" lines.

The orientations of adjoining components of the tangle diagram must be compatible.

For an oriented quantum coalgebra $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ over the field k the invariant we describe in this section is a function $\mathbf{Inv}_C : \mathbf{Tang} \longrightarrow C^*$ which is the function $\mathbf{Inv}_A$ of [9, Section 1] and [8, Section 3] when C is finite-dimensional, strict and $A = C^*$ is the dual quantum algebra. To motivate the definition of $\mathbf{Inv}_C$ we first review how $\mathbf{Inv}_A$ is constructed for oriented quantum algebras A over k. The reader is encouraged to refer to [9, Section 1] or [8, Section 3] at this point. Much of the discussion which follows parallels [11, Section 6.1].

Let $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ be an oriented quantum algebra defined over the field k and suppose that $\mathbf{T} \in \mathbf{Tang}$. We decorate each crossing of $\mathbf{T}$ according to the scheme



.

where we use the shorthand $\rho = e \otimes e'$, $\rho^{-1} = E \otimes E'$ and $(1_A \otimes t)(\rho) = e \otimes t(e')$, $(t \otimes 1_A)(\rho^{-1}) = t(E) \otimes E'$ for $t = t_{\mathsf{d}}, t_{\mathsf{u}}$. In practice we let $e \otimes e'$, $f \otimes f'$, $g \otimes g' \ldots$ denote copies of ρ and $E \otimes E'$, $F \otimes F'$, $G \otimes G' \ldots$ denote copies of ρ^{-1}.

Think of the oriented tangle as a rigid wire and think of the decorations as labeled beads which slide freely around the wire. Starting at the base point of the tangle diagram, traverse the diagram pushing the labeled beads along the wire so that the end result is a juxtaposition of labeled beads at the end point of the diagram. As a labeled bead passes through a local extrema its label is altered according to the following rules:

and

for clockwise motion;

and

for counterclockwise motion. We refer to the oriented local extrema

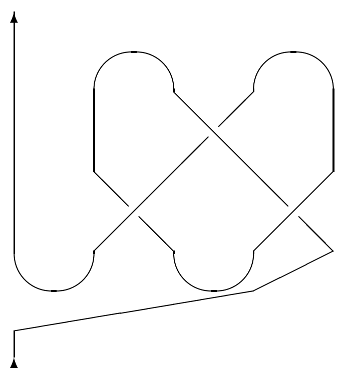
as having type (u_-), (u_+), (d_+) and (d_-) respectively. Reading the juxtaposed labeled beads in the direction of orientation results in a formal word $\mathbf{W}_A(\mathbf{T})$.

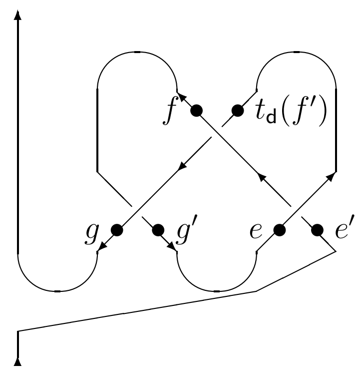
Now to define $\mathbf{W}_A(\mathbf{T})$ more formally. If $\mathbf{T}$ has no crossings then $\mathbf{W}_A(\mathbf{T}) = 1$. Suppose that $\mathbf{T}$ has $n \geq 1$ crossings. Traverse $\mathbf{T}$ in the direction of orientation and label the crossing lines $1, 2, \ldots, 2n$ in the order in which they are encountered. For $1 \leq \imath \leq 2n$ let $u_{\mathsf{d}}(\imath)$ be the number of local extrema of type (d_+) minus the number of type (d_-) encountered on the portion of the traversal from line $\imath$ to the end of the traversal of $\mathbf{T}$. We define $u_{\mathsf{u}}(\imath)$ in the same way where (u_+) and (u_-) replace (d_+) and (d_-) respectively. Then

$$\mathbf{W}_A(\mathbf{T}) = t_{\mathsf{d}}^{u_{\mathsf{d}}(1)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(1)}(x_1) \cdots t_{\mathsf{d}}^{u_{\mathsf{d}}(2n)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(2n)}(x_{2n}), \tag{8}$$

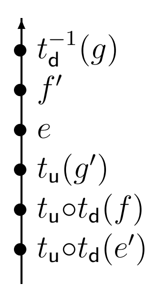
where $x_\imath$ is the decoration on the crossing line $\imath$. Replacing the formal representations of ρ and ρ^{-1} in $\mathbf{W}_A(\mathbf{T})$ by ρ and ρ^{-1} respectively we obtain an element $\mathbf{Inv}_A(\mathbf{T}) \in A$.

For example, consider the oriented 1–1 tangle diagram $\mathbf{T}_{\text{trefoil}}$ depicted below on the left.





Traversal of the 1–1 tangle diagram $\mathbf{T}_{\text{trefoil}}$ results in the juxtaposition of labeled beads



Thus

$$\mathbf{W}_A(\mathbf{T}_{\text{trefoil}}) = \left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(e')\right)\left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(f)\right)\left(t_{\mathsf{u}}(g')\right) ef' \left(t_{\mathsf{d}}^{-1}(g)\right)$$

from which we obtain after substitution

$$\mathbf{Inv}_A(\mathbf{T}_{\text{trefoil}}) = \sum_{\imath,\jmath,\ell=1}^{r} \left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\imath)\right)\left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(a_\jmath)\right)\left(t_{\mathsf{u}}(b_\ell)\right) a_\imath b_\jmath \left(t_{\mathsf{d}}^{-1}(a_\ell)\right),$$

where $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath \in A\otimes A$. Generally, the formal word $\mathbf{W}_A(\mathbf{T})$ can be viewed as merely a device which encodes instructions for defining an element of A. Since $\rho = (t\otimes t)(\rho)$ and $\rho^{-1} = (t\otimes t)(\rho^{-1})$, or symbolically $e\otimes e' = t(e)\otimes t(e')$ and $E\otimes E' = t(E)\otimes t(E')$ for $t = t_{\mathsf{d}}, t_{\mathsf{u}}$, we may introduce the rules

$$\mathbf{W}_A(\mathbf{T}) = \cdots t^p(x)\cdots t^q(y)\cdots = \cdots t^{p+\ell}(x)\cdots t^{q+\ell}(y)\cdots$$

for all integers ℓ, where $x\otimes y$ or $y\otimes x$ represents either ρ or ρ^{-1}. Thus we may rewrite

$$\mathbf{Inv}_A(\mathbf{T}_{\text{trefoil}}) = \sum_{\imath,\jmath,\ell=1}^{r} \left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\imath)\right)\left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(a_\jmath)\right)\left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\ell)\right) a_\imath b_\jmath a_\ell.$$

As a small exercise the reader is left to show that

$$\begin{aligned}
&\mathbf{Inv}_A(\mathbf{T}^{op}_{\text{trefoil}})\\
&= \sum_{\ell,\jmath,\imath=1}^{r} \left(t_{\mathsf{u}}^{-2}\circ t_{\mathsf{d}}^{-1}(a_\ell)\right)\left(t_{\mathsf{u}}^{-1}\circ t_{\mathsf{d}}^{-1}(b_\jmath)\right)\left(t_{\mathsf{u}}^{-1}\circ t_{\mathsf{d}}^{-1}(a_\imath)\right)\left(t_{\mathsf{u}}^{-1}(b_\ell)\right) a_\jmath b_\imath\\
&= \sum_{\ell,\jmath,\imath=1}^{r} a_\ell b_\jmath a_\imath \left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\ell)\right)\left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(a_\jmath)\right)\left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\imath)\right),
\end{aligned}$$

and also that

$$\mathbf{Inv}_A(\mathbf{T}_{\text{curl}}) = \sum_{\imath=1}^{r} a_\imath \left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\imath)\right), \quad \mathbf{Inv}_A(\mathbf{T}^{op}_{\text{curl}}) = \sum_{\imath=1}^{r} \left(t_{\mathsf{u}}\circ t_{\mathsf{d}}(b_\imath)\right) a_\imath.$$

Assume that A is a finite-dimensional and let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) = (A^*, b_\rho, t^*_{\mathsf{d}}, t^*_{\mathsf{u}})$ be the dual (strict) oriented quantum coalgebra. For all $\mathbf{T}\in\mathbf{Tang}$ we regard $\mathbf{Inv}_A(\mathbf{T}) \in A = A^{**} = C^*$ as a functional on C. Here we think of A as A^{**} under the identification $a(a^*) = a^*(a)$ for all $a\in A$ and $a^*\in A^*$. We set

$\mathbf{Inv}_C = \mathbf{Inv}_A$. Thus for $\mathbf{T} \in \mathbf{Tang}$ the functional $\mathbf{Inv}_C(\mathbf{T}) \in A = C^*$ is evaluated on $c \in C$ as follows. Use (8) to make the formal calculation

$$\mathbf{W}_A(\mathbf{T})(c) = c(\mathbf{W}_A(\mathbf{T})) = c_{(1)}(t_{\mathsf{d}}^{u_{\mathsf{d}}(1)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(1)}(x_1)) \cdots c_{(2n)}(t_{\mathsf{d}}^{u_{\mathsf{d}}(2n)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(2n)}(x_{2n}))$$

and replace the formal copies of ρ and ρ^{-1} by their actual values to obtain a scalar $\mathbf{Inv}_C(\mathbf{T})(c)$.

We will evaluate $\mathbf{Inv}_C(\mathbf{T}_{\text{trefoil}})(c)$ to illustrate this procedure. Recall that $b(c,d) = (c \otimes d)(\rho) = \sum_{\imath=1}^{r} c(a_\imath)d(b_\imath)$ for all $c, d \in C$. Thus we calculate, omitting the summation symbol,

$$\begin{aligned}
&\mathbf{Inv}_C(\mathbf{T}_{\text{trefoil}})(c) \\
&\quad = c\left(\left(t_{\mathsf{u}} \circ t_{\mathsf{d}}(b_\imath)\right)\left(t_{\mathsf{u}} \circ t_{\mathsf{d}}(a_\jmath)\right)\left(t_{\mathsf{u}}(b_\ell)\right) a_\imath b_\jmath \left(t_{\mathsf{d}}^{-1}(a_\ell)\right)\right) \\
&\quad = c_{(1)}\left(t_{\mathsf{u}} \circ t_{\mathsf{d}}(b_\imath)\right) c_{(2)}\left(t_{\mathsf{u}} \circ t_{\mathsf{d}}(a_\jmath)\right) c_{(3)}\left(t_{\mathsf{u}}(b_\ell)\right) c_{(4)}(a_\imath) c_{(5)}(b_\jmath) c_{(6)}\left(t_{\mathsf{d}}^{-1}(a_\ell)\right) \\
&\quad = \left(T_{\mathsf{u}} \circ T_{\mathsf{d}}(c_{(1)})(b_\imath)\right)\left(T_{\mathsf{u}} \circ T_{\mathsf{d}}(c_{(2)})(a_\jmath)\right)\left(T_{\mathsf{u}}(c_{(3)})(b_\ell)\right) c_{(4)}(a_\imath) c_{(5)}(b_\jmath)\left(T_{\mathsf{d}}^{-1}(c_{(6)})(a_\ell)\right) \\
&\quad = b(c_{(4)}, T_{\mathsf{d}} \circ T_{\mathsf{u}}(c_{(1)}))b(T_{\mathsf{d}} \circ T_{\mathsf{u}}(c_{(2)}), c_{(5)})b(T_d^{-1}(c_{(6)}), T_{\mathsf{u}}(c_{(3)}))
\end{aligned}$$

and thus

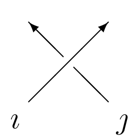
$$\mathbf{Inv}_C(\mathbf{T}_{\text{trefoil}})(c) = b(c_{(4)}, T_{\mathsf{d}} \circ T_{\mathsf{u}}(c_{(1)}))b(T_{\mathsf{d}} \circ T_{\mathsf{u}}(c_{(2)}), c_{(5)})b(T_d^{-1}(c_{(6)}), T_{\mathsf{u}}(c_{(3)}))$$

for all $c \in C$.

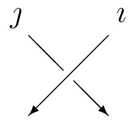
Now suppose that $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is any oriented quantum coalgebra over k. We shall define $\mathbf{Inv}_C$ in a way which agrees with our definition when C is the dual of a finite-dimensional oriented quantum algebra over k.

Let $\mathbf{T} \in \mathbf{Tang}$. If $\mathbf{T}$ has no crossings set $\mathbf{Inv}_C(\mathbf{T}) = \epsilon$. Suppose that $\mathbf{T}$ has $n \geq 1$ crossings. Starting at the base point of the tangle diagram $\mathbf{T}$, traverse $\mathbf{T}$ labeling the crossing lines of the diagram $1, \ldots, 2n$ in the order encountered. For $1 \leq \imath \leq 2n$ let $u_{\mathsf{d}}(\imath)$ and $u_{\mathsf{u}}(\imath)$ be as defined earlier in this section.

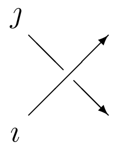
Let χ be a crossing and suppose that its over crossing and under crossing lines are labeled $\imath$ and $\jmath$ respectively. For $c \in C$ the scalar $\mathbf{Inv}_C(\mathbf{T})(c)$ is the sum of products, where each crossing contributes a factor according to:



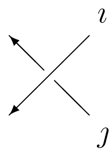
$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$



$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$

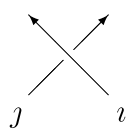


$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)+1}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$

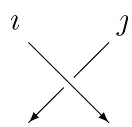


$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)+1} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$

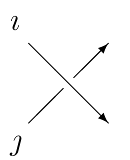
for under crossings;



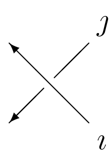
$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$



$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$



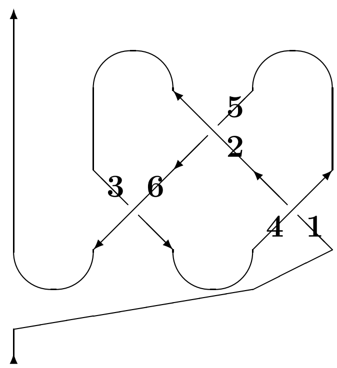
$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)+1}(c_{(\jmath)})) \cdots$$



$$\mathbf{Inv}_C(\mathbf{T})(c) = \cdots b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)+1} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \cdots$$

for over crossings. In the next section we will show that $\mathbf{Inv}_C$ determines a regular isotopy invariant of 1–1 tangle diagrams.

Let us reconsider the tangle diagram $\mathbf{T}_{\text{Trefoil}}$. Diagram traversal results in the labeling



and thus

$$\mathbf{Inv}_C(\mathbf{T}_{\text{trefoil}})(c) = b(c_{(4)}, T_{\mathsf{d}}\circ T_{\mathsf{u}}(c_{(1)}))b(T_{\mathsf{d}}\circ T_{\mathsf{u}}(c_{(2)}), c_{(5)})b(T_d^{-1}(c_{(6)}), T_{\mathsf{u}}(c_{(3)}))$$

for all $c \in C$ by the algorithm described above which agrees with our previous calculation. Observe that

$$\mathbf{Inv}_C(\mathbf{T}_{\text{curl}})(c) = b(T_{\mathsf{d}}\circ T_{\mathsf{u}}(c_1), c_{(2)}) \quad \text{and} \quad \mathbf{Inv}_C(\mathbf{T}^{op}_{\text{curl}})(c) = b(T_{\mathsf{d}}\circ T_{\mathsf{u}}(c_2), c_{(1)}).$$

Note that if $\mathbf{T}, \mathbf{T}', \mathbf{T}'' \in \mathbf{Tang}$ and $\mathbf{T}'' = \mathbf{T}\star\mathbf{T}'$ then

$$\mathbf{Inv}_C(\mathbf{T}\star\mathbf{T}') = \mathbf{Inv}_C(\mathbf{T})\mathbf{Inv}_C(\mathbf{T}'),$$

where the righthand side of the equation is the product in the dual algebra C^*.

6.2 A Proof That $\mathbf{Inv}_C$ Determines a Regular Isotopy Invariant of Oriented 1–1 Tangles

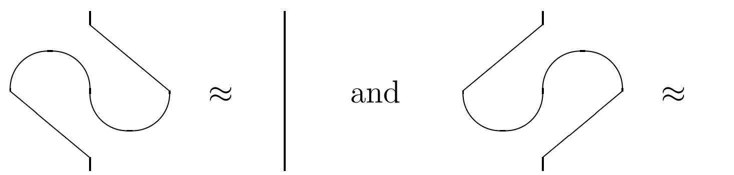
The sole purpose of this section is to show that the function $\mathbf{Inv}_C$ of Section 6.1 determines a regular isotopy invariant of oriented 1–1 tangles. This follows by:

Theorem 4 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra defined over the field k and suppose that $\mathbf{Inv}_C : \mathbf{Tang} \longrightarrow C^*$ is the function of the previous section. If $\mathbf{T}, \mathbf{T}' \in \mathbf{Tang}$ are regularly isotopic then $\mathbf{Inv}_C(\mathbf{T}) = \mathbf{Inv}_C(\mathbf{T}')$.*

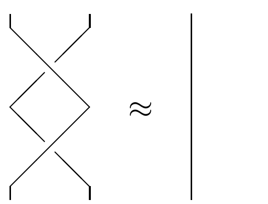
Proof: The reader will find a discussion of regular isotopy, which we assume as background material, in many references. Here we follow the conventions of [4].

The regular isotopy equivalences are

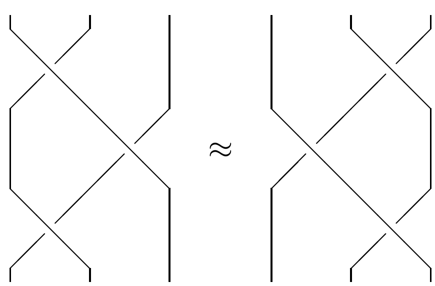
(M.1)



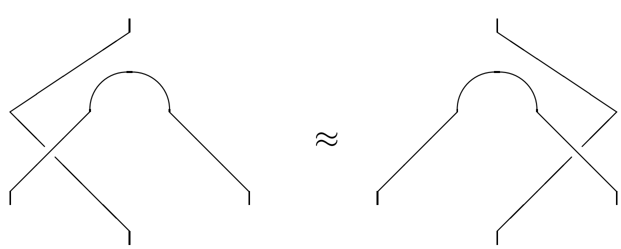
(M.2)



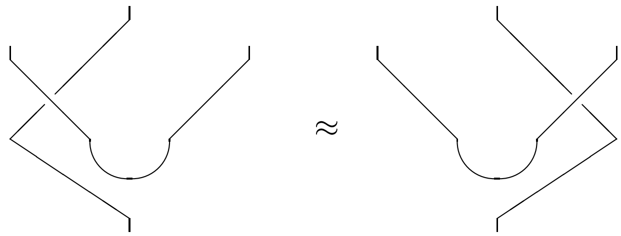
(M.3)



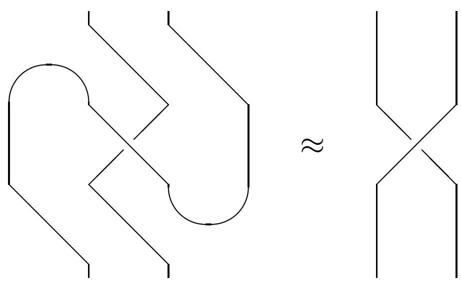
(M.4)



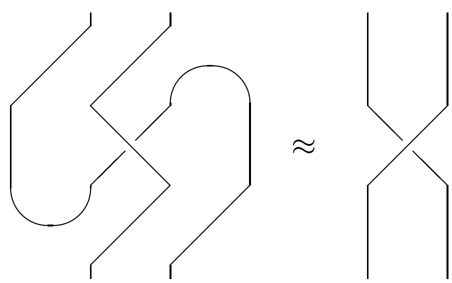
and



and (M.2rev)–(M.4rev), which are (M.2)–(M.4) respectively with over crossings replaced by under crossings and vice versa. The "twist moves"



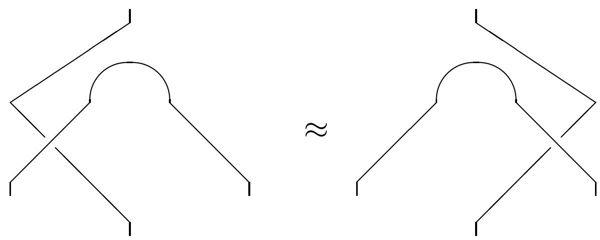
and



are consequences of (M.1), (M.2) and (M.4). These are important in that they allow for crossing types to be changed.

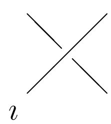
Let $\mathbf{T}, \mathbf{T}' \in \mathbf{Tang}$ and suppose that a part of $\mathbf{T}$ is the figure on the left in one of the equivalences of (M.1)–(M.5) or (M.2rev)–(M.5rev) and that $\mathbf{T}'$ is obtained from $\mathbf{T}$ by replacing the figure on the left with the figure on the right. To prove the theorem we need only show that $\mathbf{Inv}_C(\mathbf{T}) = \mathbf{Inv}_C(\mathbf{T}')$. There are many cases to consider since all possible orientations must be taken into account. We will carefully analyze the typical cases, leaving the remainder for the reader to work out. Let u'_{d} and u'_{u} be the counterparts of u_{d} and u_{u} respectively for $\mathbf{T}'$.

Consider the first equivalence

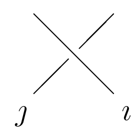


(9)

of (M.4). In this case



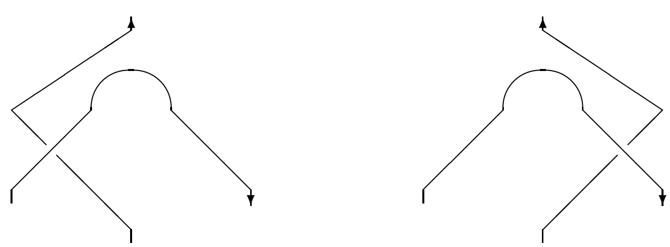
in $\mathbf{T}$ is replaced by



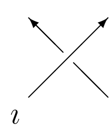
in $\mathbf{T}'$.

There are four possible orientations associated with (9).

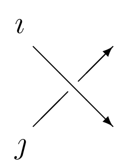
Case M.4.1:



In this case



in $\mathbf{T}$ is replaced by

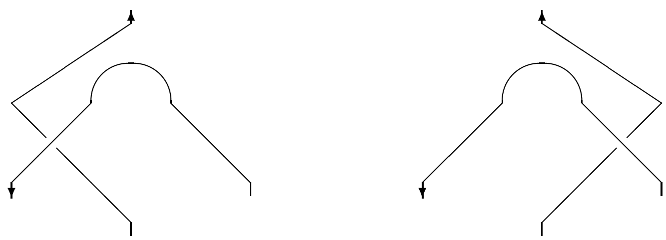


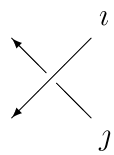
in $\mathbf{T}'$. Observe that u'_{d} and u'_{u} agree with u_{d} and u_{u} respectively with the exception $u'_{\mathsf{u}}(\imath) = u_{\mathsf{u}}(\imath) + 1$. Since $T_{\mathsf{d}}, T_{\mathsf{u}}$ commute and (qc.2) holds for b it follows that

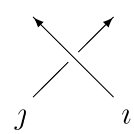
$$\begin{aligned}
&b(T_{\mathsf{d}}^{u'_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u'_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u'_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u'_{\mathsf{u}}(\jmath)+1}(c_{(\jmath)})) \\
&\quad = \quad b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)+1}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)+1}(c_{(\jmath)})) \\
&\quad = \quad b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)}))
\end{aligned}$$

The contributions which the other crossings of $\mathbf{T}$ make to $\mathbf{Inv}_C(\mathbf{T})(c)$ are unaffected by the replacement of the figure on the left in (9) with the right on the right. Therefore $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in this case.

Case M.4.2:



In this case 

in $\mathbf{T}$ is replaced by 

in $\mathbf{T}'$. Observe that u'_{d} and u'_{u} agree with u_{d} and u_{u} respectively with the exception $u'_{\mathsf{d}}(\imath) = u_{\mathsf{d}}(\imath) + 1$. Since

$$\begin{aligned} & b^{-1}(T_{\mathsf{d}}^{u'_{\mathsf{d}}(\imath)} \circ T_{\mathsf{u}}^{u'_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u'_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u'_{\mathsf{u}}(\jmath)}(c_{(\jmath)})) \\ = \;& b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)+1} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)} \circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)}) \end{aligned}$$

and the contributions which the other crossings of $\mathbf{T}$ make to $\mathbf{Inv}_C(\mathbf{T})(c)$ are unaffected by the replacement of the figure on the left in (9) with the right on the right, $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in this case.

The calculations in the other two cases, which are Cases M.4.1 and M.4.2 with orientations reversed, are similar to those in Cases M.4.1 and M.4.2 respectively. Thus $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ when $\mathbf{T}$ is altered according to (9).

By a similar argument it follows that $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ when $\mathbf{T}$ is altered according to the second equivalence of (M.4) and, since (qc.2) holds for b^{-1} also, $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ when $\mathbf{T}$ is altered according to (M.4rev).

It is clear that $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ when $\mathbf{T}$ is altered according to (M.1). The following non-standard notation for the coproduct

$$\begin{aligned} \Delta^{(m-1)}(c) &= c_{(1)} \otimes \cdots \otimes c_{(\ell)} \otimes c_{(\ell+1)} \otimes \cdots \otimes c_{(\ell')} \otimes c_{(\ell'+1)} \cdots \otimes c_{(m)} \\ &= c_{(1)} \otimes \cdots \otimes c_{(\ell)(1)} \otimes c_{(\ell)(2)} \otimes \cdots \otimes c_{(\ell')(1)} \otimes c_{(\ell')(2)} \cdots \otimes c_{(m)} \end{aligned}$$

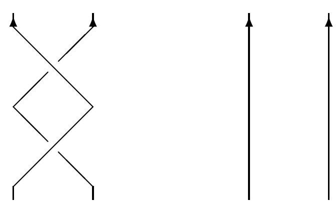
will be very useful in our analysis of (M.2). To emphasize, in the second expression for $\Delta^{(m-1)}$ differs from the first *only* in that the subscripts $(\ell), (\ell{+}1)$ are replaced by $(\ell)(1), (\ell)(2)$ and that $(\ell'), (\ell'{+}1)$ are replaced by $(\ell')(1), (\ell')(2)$. Likewise the non-standard notation

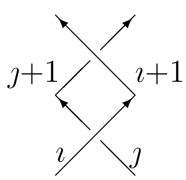
$$\begin{aligned}
&\Delta^{(m-1)}(c)\\
&\quad = \ c_{(1)}\otimes\cdots\otimes c_{(\ell)}\otimes c_{(\ell+1)}\cdots\otimes c_{(\ell')}\otimes c_{(\ell'+1)}\cdots\otimes c_{(\ell'')}\otimes c_{(\ell''+1)}\cdots\otimes c_{(m)}\\
&\quad = \ c_{(1)}\otimes\cdots\otimes c_{(\ell)(1)}\otimes c_{(\ell)(2)}\cdots\otimes c_{(\ell')(1)}\otimes c_{(\ell')(2)}\cdots\otimes c_{(\ell'')(1)}\otimes c_{(\ell'')(2)}\cdots\otimes c_{(m)}
\end{aligned}$$

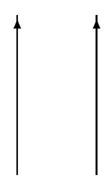
will be very useful in our analysis of our analysis of (M.3). These manipulations with the subscripts are justified the coassociativity of the coproduct.

We consider (M.2) next. There are four cases to analyze.

Case M.2.1:



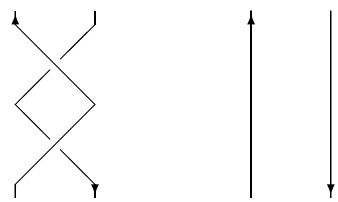
In this case 

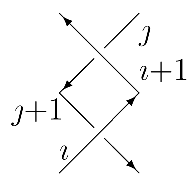
in $\mathbf{T}$ is replaced by  in $\mathbf{T}'$. Since b^{-1} is right inverse of b and $T_{\mathsf{d}}, T_{\mathsf{u}}$ are commuting coalgebra automorphisms of C with respect to $\{b, b^{-1}\}$, it follows by part a) of Lemma 1 that the contribution which the two crossings above make to the calculation of $\mathbf{Inv}_C(\mathbf{T})(c)$ is

$$\begin{aligned}
&b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)}))b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath+1)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath+1)}))\\
&\quad = \ b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)(1)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)(1)}))b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)(2)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)(2)}))\\
&\quad = \ b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)})_{(1)}, T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})_{(1)})b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)})_{(2)}, T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})_{(2)})\\
&\quad = \ \epsilon(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}))\epsilon(T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\imath')}))\\
&\quad = \ \epsilon(c_{(\imath)})\epsilon(c_{(\jmath)})\\
&\quad = \ \epsilon(c_{(\imath)(1)})\epsilon(c_{(\imath)(2)})\epsilon(c_{(\jmath)(1)})\epsilon(c_{(\jmath)(2)})\\
&\quad = \ \epsilon(c_{(\imath)})\epsilon(c_{(\imath+1)})\epsilon(c_{(\jmath)})\epsilon(c_{(\jmath+1)}).
\end{aligned}$$

Since the contributions which the other crossings of $\mathbf{T}$ make to $\mathbf{Inv}_C(\mathbf{T})(c)$ are unaffected by the replacement of the figure on the left in Case M.2.1 with the right on the right, $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in this case.

Case M.2.2:



In this case 

in $\mathbf{T}$ is replaced by in $\mathbf{T}'$. Since $T_{\mathsf{d}}, T_{\mathsf{u}}$ are commuting coalgebra automorphisms of C with respect to $\{b, b^{-1}\}$, it follows by part b) of Lemma 1 and the second equation of (qc.1) that the contribution which the two crossings above make to the calculation of $\mathbf{Inv}_C(\mathbf{T})(c)$ is

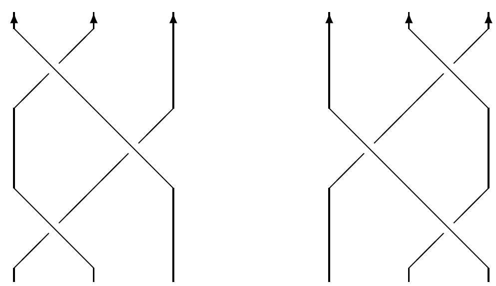
$$\begin{aligned}
&b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)+1}(c_{(\imath)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)}))b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath+1)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)+1}(c_{(\jmath)}))\\
&= b^{-1}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)+1}(c_{(\imath)(1)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)(2)})) \times\\
&\qquad b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)(2)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)+1}(c_{(\jmath)(1)}))\\
&= b^{-1}(T_{\mathsf{u}}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)})_{(1)}), T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})_{(2)}) \times\\
&\qquad b(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)})_{(2)}, T^{\mathsf{u}}(T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\jmath)})_{(1)})))\\
&= \epsilon(T^{u(\imath)}(c_{(\imath)}))\epsilon(T^{u(\imath')}(c_{(\imath')}))\\
&= \epsilon(T_{\mathsf{d}}^{u_{\mathsf{d}}(\imath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\imath)}(c_{(\imath)}))\epsilon(T_{\mathsf{d}}^{u_{\mathsf{d}}(\jmath)}\circ T_{\mathsf{u}}^{u_{\mathsf{u}}(\jmath)}(c_{(\imath')}))\\
&= \epsilon(c_{(\imath)})\epsilon(c_{(\jmath)})\\
&= \epsilon(c_{(\imath)(1)})\epsilon(c_{(\imath)(2)})\epsilon(c_{(\jmath)(1)})\epsilon(c_{(\jmath)(2)})\\
&= \epsilon(c_{(\imath)})\epsilon(c_{(\imath+1)})\epsilon(c_{(\jmath)})\epsilon(c_{(\jmath+1)}).
\end{aligned}$$

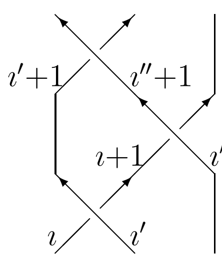
Since the contributions which the other crossings of $\mathbf{T}$ make to $\mathbf{Inv}_C(\mathbf{T})(c)$ are unaffected by the replacement of the figure on the left in Case M.2.1 with the right on the right, $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in this case.

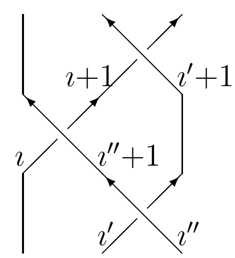
Using the fact that b^{-1} is a left inverse for b the argument for Cases M.2.1 is easily modified to show that $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in Case M.2.3, which is Case M.2.1 with orientations reversed. Using the first equation of (qc.1) the argument for Cases M.2.2 is easily modified to show that $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in Case M.2.4, which is Case M.2.2 with orientations reversed.

It remains to analyze (M.3). We consider the possible orientations of the lines of the figures described in (M.3), reading left to right.

Case M.3.1



In this case 

in $\mathbf{L}$ is replaced by 

in $\mathbf{L}'$. Since $u(\imath{+}1) = u(\imath)$, $u(\imath'{+}1) = u(\imath')$ and $u(\imath''{+}1) = u(\imath'')$, the contribution which the figure on the left above makes to the calculation of $\mathbf{Inv}_C(\mathbf{T})(c)$ is

$$b^{-1}(T^{u(\imath'')}(c_{(2)}), T^{u(\imath')}(d_{(2)}))b^{-1}(T^{u(\imath'')}(c_{(1)}), T^{u(\imath)}(e_{(2)}))b^{-1}(T^{u(\imath')}(d_{(1)}), T^{u(\imath)}(e_{(1)}))$$

and the contribution which the figure on the right above makes to the calculation of $\mathbf{Inv}_C(\mathbf{T}')(c)$ is

$$b^{-1}(T^{u(\imath'')}(c_{(1)}), T^{u(\imath')}(d_{(1)}))b^{-1}(T^{u(\imath'')}c_{(2)}, T^{u(\imath)})(e_{(1)}))b^{-1}(T^{u(\imath')}d_{(2)}, T^{u(\imath)})(e_{(2)}))$$

where $c = c_{(\imath'')}$, $d = c_{(\imath'')}$, and $e = c_{(\imath)}$. By part a) of Lemma 1 the two contributions are the same if

$$\begin{aligned} & b^{-1}(c_{(2)}, T^v(d_{(2)}))b^{-1}(c_{(1)}, e_{(1)})b^{-1}(T^v(d_{(1)}), e_{(2)}) \\ = \; & b^{-1}(c_{(1)}, T^v(d_{(1)}))b^{-1}(c_{(2)}, e_{(1)})b^{-1}(T^v(d_{(2)}), e_{(2)}) \end{aligned}$$

for all $c, d, e \in C$. Since b^{-1} satisfies (qc.2), using part a) of Lemma 1 again we see that this last equation holds if and only if

$$\begin{aligned} & b^{-1}(T^{-v}(c_{(2)}), d_{(2)})b^{-1}(T^{-v}(c_{(1)}), T^{-v}(e_{(1)}))b^{-1}(d_{(1)}, T^{-v}(e_{(2)})) \\ = \; & b^{-1}(T^{-v}(c_{(1)}), d_{(1)})b^{-1}(T^{-v}(c_{(2)}), T^{-v}(e_{(1)}))b^{-1}(T^{v}(d_{(2)}), e_{(2)}) \end{aligned}$$

holds for all $c, d, e \in C$ if and only if

$$\begin{aligned} & b^{-1}(T^{-v}(c)_{(2)}, d_{(2)})b^{-1}(T^{-v}(c)_{(1)}, T^{-v}(e)_{(1)})b^{-1}(d_{(1)}, T^{-v}(e)_{(2)}) \\ = \; & b^{-1}(T^{-v}(c)_{(1)}, d_{(1)})b^{-1}(T^{-v}(c)_{(2)}, T^{-v}(e)_{(2)})b^{-1}(d_{(2)}, T^{-v}(e)_{(1)})) \end{aligned}$$

holds for all $c, d, e \in C$ which in turn holds if and only if

$$\begin{aligned} & b^{-1}(c_{(2)}, d_{(2)})b^{-1}(c_{(1)}, e_{(1)})b^{-1}(d_{(1)}, e_{(2)}) \\ = \; & b^{-1}(c_{(1)}, d_{(1)})b^{-1}(c_{(2)}, e_{(1)})b^{-1}(d_{(2)}, e_{(2)}) \end{aligned}$$

holds for all $c, d, e \in C$. The last equation is (qc.3) for b^{-1} which holds since (qc.3) holds for b. Thus the two contributions are the same. Since the contributions which the other crossings of $\mathbf{T}$ make to $\mathbf{Inv}_C(\mathbf{T})(c)$ are unaffected by the replacement of the figure on the left in Case M.3.1 with the right on the right, $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ in Case M.3.1.

Using similar arguments one can show that $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ for all $c \in C$ in Case M.3.2 (up up down) if

$$\begin{aligned} & b^{-1}(c_{(2)}, d_{(2)})b(c_{(1)}, e_{(1)})b(d_{(1)}, e_{(2)}) \\ = \; & b^{-1}(c_{(1)}, d_{(1)})b(c_{(2)}, e_{(2)})b(d_{(2)}, e_{(1)}) \end{aligned} \tag{10}$$

for all $c, d, e \in C$, in Case M.3.3 (up down up) if

$$\begin{aligned} & b(c_{(2)}, d_{(1)})b^{-1}(c_{(1)}, e_{(2)})b(d_{(2)}, T^{2}(e_{(1)})) \\ = \; & b(c_{(1)}, d_{(2)})b^{-1}(c_{(2)}, e_{(1)})b(d_{(1)}, T^{2}(e_{(2)})) \end{aligned} \tag{11}$$

for all $c, d, e \in C$, and in Case M.3.4 (up down down) if

$$\begin{aligned} & b(c_{(2)}, d_{(1)})b(c_{(1)}, e_{(1)})b^{-1}(d_{(2)}, e_{(2)}) \\ = \; & b(c_{(1)}, d_{(2)})b(c_{(2)}, e_{(2)})b^{-1}(d_{(1)}, e_{(1)}) \end{aligned} \tag{12}$$

for all $c, d, e \in C$. Cases M.2.5–M.3.8, which are Cases M.3.1–M.3.4 with orientations reversed, reduce to Cases M.3.1–M.3.4. Cases M.3rev.1–M.3rev.8

are Cases M.3.1–M.3.8 for the oriented quantum coalgebra (C, b^{-1}, T^{-1}). Thus to complete the proof of the theorem we need only establish (10)–(12).

To establish (10) we define linear maps $\ell, r, u : C\otimes C\otimes C \longrightarrow k$ by

$$\ell(c\otimes d\otimes e) = b^{-1}(c_{(2)}, d_{(2)})b(c_{(1)}, e_{(1)})b(d_{(1)}, e_{(2)}),$$

$$r(c\otimes d\otimes e) = b^{-1}(c_{(1)}, d_{(1)})b(c_{(2)}, e_{(2)})b(d_{(2)}, e_{(1)})$$

and

$$u(c\otimes d\otimes e) = b(c, d)\epsilon(e)$$

for all $c, d, e \in C$. Since u is invertible in the dual algebra $(C\otimes C\otimes C)^*$ and $u\ell u = uru$, we conclude that $\ell = r$, which is to say that (10) holds.

To establish (12) we define linear maps $\ell, r, u : C\otimes C\otimes C \longrightarrow k$ by

$$\ell(c\otimes d\otimes e) = b(c_{(2)}, d_{(1)})b(c_{(1)}, e_{(1)})b^{-1}(d_{(2)}, e_{(2)}),$$

$$r(c\otimes d\otimes e) = b(c_{(1)}, d_{(1)})b(c_{(2)}, e_{(2)})b^{-1}(d_{(1)}, e_{(1)})$$

and

$$u(c\otimes d\otimes e) = \epsilon(e)b(d, e)$$

for all $c, d, e \in C$. Again, u is invertible in the dual algebra $(C\otimes C\otimes C)^*$ and again $u\ell u = uru$. Thus $\ell = r$, or equivalently (10) holds.

Equation (11) is perhaps the most interesting of (10)–(12). Since T^{-1} is a coalgebra automorphism of C with respect to $\{b, b^{-1}\}$ by part a) of Lemma 1, the equations of (qc.1) can be reformulated

$$b(c_{(1)}, T^2(d_{(2)}))b^{-1}(c_{(2)}, d_{(1)}) = \epsilon(c)\epsilon(d)$$

and

$$b^{-1}(c_{(1)}, d_{(2)})b(c_{(2)}, T^2(d_{(1)})) = \epsilon(c)\epsilon(d)$$

for all $c.d \in C$. Thus the right hand side of the equation of (11)

$$\begin{aligned}
&b(c_{(1)}, d_{(2)})b^{-1}(c_{(2)}, e_{(1)})b(d_{(1)}, T^2(e_{(2)}))\\
&\quad= \quad b^{-1}(c_{(1)}, e_{(4)})b(c_{(3)}, d_{(2)})b^{-1}(c_{(4)}, e_{(1)})b(d_{(1)}, T^2(e_{(2)})b(c_{(2)}, T^2(e_{(3)}))\\
&\quad= \quad b^{-1}(c_{(1)}, e_{(4)})b(c_{(2)}, d_{(1)})b^{-1}(c_{(4)}, e_{(1)})b(d_{(2)}, T^2(e_{(3)})b(c_{(3)}, T^2(e_{(2)}))\\
&\quad= \quad b^{-1}(c_{(1)}, e_{(2)})b(c_{(2)}, d_{(1)})b(d_{(2)}, T^2(e_{(1)}))\\
&\quad= \quad b(c_{(2)}, d_{(1)})b^{-1}(c_{(1)}, e_{(2)})b(d_{(2)}, T^2(e_{(1)}))
\end{aligned}$$

is equal to the left hand side. We have established (11) which completes the proof of the theorem. $\square$

Apropos of the proof of Theorem 4, observe that (10) and (12) reduce to the finite-dimensional case since C is the sum of its finite-dimensional subcoalgebras. For suppose that C is finite-dimensional, $b : C\otimes C \longrightarrow k$ is a bilinear form which is invertible and satisfies (qc.3). Let $R \in C^*\otimes C^*$ be defined by $b(c, d) = R(c\otimes d)$ for all $c, d \in C$. Then R is invertible and $R_{1\,2}R_{1\,3}R_{2\,3} = R_{2\,3}R_{1\,3}R_{1\,2}$. Equations (10) and (12) translate to $R^{-1}_{1\,2}R_{2\,3}R_{1\,3} = R_{1\,3}R_{2\,3}R^{-1}_{1\,2}$ and $R^{-1}_{2\,3}R_{1\,2}R_{1\,3} = R_{1\,3}R_{1\,2}R^{-1}_{2\,3}$ respectively which are consequences of the preceding equation. Also, once $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ for all $c \in C$ is established in the cases for (M.2), (M.4) and in Case M.3.1, necessarily $\mathbf{Inv}_C(\mathbf{T})(c) = \mathbf{Inv}_C(\mathbf{T}')(c)$ for all $c \in C$ in Case M.4.3 for topological reasons.

To calculate the invariant $\mathbf{Inv}_C$ we need only consider standard oriented quantum coalgebras.

Theorem 5 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k. Then $\mathbf{Inv}(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})(\mathbf{T}) = \mathbf{Inv}_{(C,b,1_C,T_{\mathsf{d}}\circ T_{\mathsf{u}})}(\mathbf{T})$ for all $\mathbf{T} \in \mathbf{Tang}$.*

PROOF: We may assume $\mathbf{T}$ has a crossing and that all of of its crossings are oriented upward. The result now follows as $u_{\mathsf{u}}(\imath) = u_{\mathsf{d}}(\imath)$ for all lines $\imath$ of $\mathbf{T}$. See the discussion preceding [8, Proposition 3]. $\square$

The invariants $\mathbf{Inv}_C$ and $\mathbf{Inv}_{C^{cop}}$ have a very natural relationship.

Lemma 3 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over k. Then $\mathbf{Inv}_C(\mathbf{T}^{op}) = \mathbf{Inv}_{C^{cop}}(\mathbf{T})$ for all $\mathbf{T} \in \mathbf{Tang}$.*

PROOF: We may assume that $\mathbf{T}$ has $n \geq 1$ crossings. A crossing line of $\mathbf{T}$ which has label $\imath$ in $\mathbf{T}^{op}$ has label $n + \imath - 1$ in $\mathbf{T}$. Let s be the sum of the local extrema of $\mathbf{T}$ which are oriented counter clockwise minus the number oriented clockwise. Then $s = u(n + \imath - 1) - u^{op}(\imath)$, or equivalently $u^{op}(\imath) = u(n + \imath - 1) - s$, for all $1 \leq \imath \leq n$. Thus

$$\begin{aligned}
\mathbf{Inv}_C(\mathbf{T}^{op})(c) &= \ldots b^r(T^{u^{op}}(c_{(\imath)}), T^{u^{op}}(c_{(\jmath)}))\ldots \\
&= \ldots b^r(T^{u(n+\imath-1)-s}(c_{(\imath)}), T^{u(n+\jmath-1)-s}(c_{(\jmath)}))\ldots \\
&= \ldots b^r(T^{u(n+\imath-1)}(c_{(\imath)}), T^{u(n+\jmath-1)}(c_{(\jmath)}))\ldots \\
&= \mathbf{f}_{C^{cop}}(\mathbf{T})(c)
\end{aligned}$$

for all $c \in C$, where $r = \pm 1$. $\square$

7 Oriented 1–1 Tangle Invariants Arising from Cocommutative Oriented Quantum Coalgebras

Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over k and suppose that C is cocommutative. To compute $\mathbf{Inv}_C$ we may assume that $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) = (C, b, 1_C, T)$ is standard by Theorem 5. Since C is cocommutative it follows by (qc.1) that b and the bilinear form $b' : C \times C \longrightarrow k$ defined by $b'(c, d) = b(c, T(d))$ for all $c, d \in C$ are both inverses for b^{-1}. Therefore $b' = b$, and using (qc.2) we deduce

$$b(c, T(d)) = b(c, d) = b(T(c), d) \tag{13}$$

for all $c, d \in C$. Since T is a coalgebra automorphism of C with respect to $\{b, b^{-1}\}$ it follows by (13) that (13) holds for b^{-1} and T as well; therefore

$$b^r(T^u(c), T^v(d)) = b(c, d) \tag{14}$$

for all integers u, v and $c, d \in C$, where $r = \pm 1$.

Suppose that $\mathbf{T} \in \mathbf{Tang}$ is an oriented 1–1 tangle diagram with $n \geq 1$ crossings. Let $c \in C$. Since C is cocommutative $\Delta^{(2n-1)}(c) = c_{(\imath_1)} \otimes \cdots \otimes c_{(\imath_{2n})}$, where $\imath_1, \ldots, \imath_{2n}$ is any arrangement of $1, \ldots, 2n$; see [11, Section 7.4] for example. This last equation and (14) show that any crossing χ of $\mathbf{T}$ with crossing lines labeled $\imath$ and $\jmath$ contributes the factor $b^{\mathrm{sign}\,\chi}(c_{(\imath)}, c_{(\jmath)})$ to the formulation of $\mathbf{Inv}_C(\mathbf{T})(c)$.

We follow [4] in our convention for the sign of an oriented crossing. The sign of an oriented crossing is 1 if as the under crossing line is traversed in the direction of orientation the direction of the over crossing line is to the right, otherwise the sign of the crossing is -1. The writhe of an oriented 1–1 tangle diagram, denoted by $\mathrm{writhe}\,\mathbf{T}$, is 0 if the tangle has no crossings; otherwise the writhe is defined to be the sum of the signs of the crossings.

Now let $\chi_1, \ldots, \chi_n$ be the crossings of $\mathbf{T}$. Using the cocommutativity of C again we may thus write

$$\begin{aligned}
\mathbf{Inv}_C(\mathbf{T})(c) &= b^{\mathrm{sign}\,\chi_1}(c_{(1)(1)}, c_{(2)(1)}) \cdots b^{\mathrm{sign}\,\chi_n}(c_{(1)(n)}, c_{(2)(n)}) \\
&= b_{(\ell)}^{\mathrm{sign}\,\chi_1}(c_{(1)(1)})(c_{(2)(1)}) \cdots b_{(\ell)}^{\mathrm{sign}\,\chi_n}(c_{(1)(n)})(c_{(2)(n)}) \\
&= \left(b_{(\ell)}^{\mathrm{sign}\,\chi_1}(c_{(1)(1)}) \cdots b_{(\ell)}^{\mathrm{sign}\,\chi_n}(c_{(1)(n)})\right)(c_{(2)})
\end{aligned}$$

$$
\begin{aligned}
&= b_{(\ell)}^{\operatorname{sign}\chi_1+\cdots+\operatorname{sign}\chi_n}(c_{(1)})(c_{(2)}) \\
&= b_{(\ell)}^{\operatorname{writhe}\mathbf{T}}(c_{(1)})(c_{(2)}) \\
&= b^{\operatorname{writhe}\mathbf{T}}(c_{(1)}, c_{(2)}).
\end{aligned}
$$

With the convention $b^0(c,d) = \epsilon(c)\epsilon(d)$ for all $c, d \in C$, we conclude that

$$
\mathbf{Inv}_C(\mathbf{T})(c) = b^{\operatorname{writhe}\mathbf{T}}(c_{(1)}, c_{(2)}) \tag{15}
$$

for all $\mathbf{T} \in \mathbf{Tang}$ and $c \in C$. Thus the regular isotopy invariant writhe of oriented 1–1 tangle diagrams dominates $\mathbf{Inv}_C$, meaning that whenever $\mathbf{T}, \mathbf{T}' \in \mathbf{Tang}$ satisfy $\operatorname{writhe}\mathbf{T} = \operatorname{writhe}\mathbf{T}'$ then $\mathbf{Inv}_C(\mathbf{T}) = \mathbf{Inv}_C(\mathbf{T}')$.

8 Regular Isotopy Invariants of Oriented Knots and Links Which Arise from a Twist Oriented Quantum Coalgebra

Throughout this section $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}}, G)$ is a twist oriented quantum coalgebra over k; that is $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is a strict oriented quantum coalgebra over k and $G \in C^*$ in an invertible element which satisfies $T_{\mathsf{d}}^*(G) = T_{\mathsf{u}}^*(G) = G$ and $T_{\mathsf{d}} \circ T_{\mathsf{u}}(c) = G^{-1} \rightharpoonup c \leftharpoonup G$ for all $c \in C$. The notion of twist quantum coalgebra is introduced in [7, Section 4]. Note that $(C^{cop}, b, T_{\mathsf{d}}, T_{\mathsf{u}}, G^{-1})$ is a twist oriented quantum coalgebra as well.

We represent oriented knots and links as diagrams in the plane with respect to the vertical direction. Let $\mathcal{K}$ be the set of oriented knot diagrams and $\mathcal{L}$ be the set of oriented link diagrams with respect to the vertical direction. We will show T-invariant cocommutative elements $\mathsf{c} \in C$ give rise to scalar valued functions $\mathbf{f}_{C,\mathsf{c}} : \mathcal{L} \longrightarrow k$ which are constant on the regular isotopy classes of oriented link diagrams (and thus $\mathbf{f}_{C,\mathsf{c}}$ defines a regular isotopy invariant of oriented knots and links). The function $\mathbf{f}_{C,\mathsf{c}}$ restricted to the set of oriented knot diagrams $\mathcal{K}$ is closely related to the function $\mathbf{Inv}_C$ of Section 6.1.

A very important example of a cocommutative element is the trace function $\mathrm{Tr} : \mathrm{M}_n(k) \longrightarrow k$ which we regard as an element of $\mathrm{C}_n(k) = \mathrm{M}_n(k)^*$. Since any algebra automorphism t of $\mathrm{M}_n(k)$ is described by $t(x) = GxG^{-1}$

for all $x \in \mathrm{M}_n(k)$, where $G \in \mathrm{M}_n(k)$ is invertible, it follows that Tr is $T_{\mathsf{d}}, T_{\mathsf{u}}$-invariant for all twist oriented quantum coalgebra structures $(\mathrm{C}_n(k), b, T_{\mathsf{d}}, T_{\mathsf{u}})$ on $\mathrm{C}_n(k)$. See the corollary to [2, Theorem 4.3.1].

8.1 The Function $\mathbf{f}_{C,\mathsf{c}}$ Defined on Oriented Knot Diagrams

Let c be a $T_{\mathsf{d}} \circ T_{\mathsf{u}}$-invariant cocommutative element of C and suppose that $\mathbf{K} \in \mathcal{K}$. To define the scalar $\mathbf{f}_{C,\mathsf{c}}(\mathbf{K})$ we first construct a functional $\mathbf{f} \in C^*$ as follows. If $\mathbf{K}$ has no crossings set $\mathbf{f} = \epsilon$.

Suppose that $\mathbf{K}$ has $n \geq 1$ crossings. Choose a point P on a vertical line in the knot diagram $\mathbf{K}$. (There is no harm, under regular isotopy, in inserting a vertical line at the end of a crossing line or local extrema – thus we may assume that $\mathbf{K}$ has a vertical line.) We refer to our chosen point P as the starting point.

Traverse the knot diagram $\mathbf{K}$, starting at P and moving in the direction of the orientation, labelling the crossing lines $1, \ldots, 2n$ in the order encountered. For $c \in C$ let $\mathbf{f}(c)$ be a sum of products, where each crossing contributes a factor by the same algorithm which was used to describe $\mathbf{Inv}_C(\mathbf{T})(c)$ in Section 6.1. The proof of Theorem 4 can be repeated verbatim to show that $\mathbf{f}(c)$ is unaffected by the replacement of local parts of the knot diagram $\mathbf{K}$ by their equivalents according to (M.1)–(M.5) and (M.2rev)–(M.5rev).

Let d be the Whitney degree of the oriented knot diagram $\mathbf{K}$. Then $2d$ is the number of local extrema with clockwise orientation minus the number of extrema with counterclockwise orientation. We will show that the scalar

$$(G^d\mathbf{f})(\mathsf{c}) = G^d(\mathsf{c}_{(1)})\mathbf{f}(\mathsf{c}_{(2)})$$

does not depend on the starting point P. Observe to calculate $\mathbf{f}$ we may assume that all crossings are oriented in the upright position. Altering $\mathbf{K}$ to achieve this will not change the Whitney degree. Thus we may assume that all crossings are oriented in the upright position. In light of the proof of Theorem 5 we may also assume that $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is standard. Set $T = T_{\mathsf{u}}$.

Consider a new starting point P_{new} which precedes P in the orientation of $\mathbf{K}$ and has the property that traversal of the portion of the diagram $\mathbf{K}$ form P_{new} to P in the direction of the orientation passes through exactly one local extremum. Let $\mathbf{f}_{new}$ be the analog of $\mathbf{f}$ constructed for P_{new} and let

$m+1, \ldots, 2n$ be the labels of the crossing lines between P_{new} and P. Set $r=1$ if the extremum which precedes P has clockwise orientation and set $r=-1$ otherwise. Then

$$G^d(\mathsf{c}_{(1)})\mathbf{f}(\mathsf{c}_{(2)}) = G^d(\mathsf{c}_{(1)})\mathbf{f}_{new}(\mathsf{c}_{(2)})$$

if

$$\begin{aligned} & G^d(\mathsf{c}_{(1)})T^{\ell_1}(\mathsf{c}_{(2)(1)})\otimes\cdots\otimes T^{\ell_m}(\mathsf{c}_{(2)(m)})\otimes\mathsf{c}_{(2)(m+1)}\otimes\cdots\otimes\mathsf{c}_{(2)(2n)} \\ = \;& G^d(\mathsf{c}_{(1)})T^{\ell_1+r}(\mathsf{c}_{(2)(2n-m+1)})\otimes\cdots\otimes T^{\ell_m+r}(\mathsf{c}_{(2)(2n)})\otimes \\ & T^{-2d+r}(\mathsf{c}_{(2)(1)})\otimes\cdots\otimes T^{-2d+r}(\mathsf{c}_{(2)(2n-m)}) \end{aligned} \tag{16}$$

for all integers $\ell_1, \ldots, \ell_m$. We will establish (16) by showing for all $a_1, \ldots, a_{2n} \in C^*$ that $a_1\otimes\cdots\otimes a_{2n}$ applied to both sides of the equation of (16) gives the same result.

Now $t = T^*$ is an algebra automorphism of C^* since T is a coalgebra automorphism of C. The axioms $T^*(G) = G$ and $T(c) = G^{-1}\rightharpoonup c \leftharpoonup G$ for all $c \in C$ translate to $t(G) = G$ and $t(a) = GaG^{-1}$ for all $a \in C^*$. Since c is cocommutative $ab(\mathsf{c}) = ba(\mathsf{c})$ for all $a, b \in C^*$. Let $a_1, \ldots, a_{2n} \in C^*$. Applying $a_1\otimes\cdots\otimes a_{2n}$ to the righthand side of the equation of (16) gives

$$\begin{aligned} & G^d t^{-2d+r}(a_{m+1})\cdots t^{-2d+r}(a_{2n})t^{\ell_1+r}(a_1)\cdots t^{\ell_m+r}(a_m)(\mathsf{c}) \\ = \;& t^r(G^d t^{-2d}(a_{m+1}\cdots a_{2n})t^{\ell_1}(a_1)\cdots t^{\ell_m}(a_m))(\mathsf{c}) \\ = \;& G^d G^{-d} a_{m+1}\cdots a_{2n} G^d t^{\ell_1}(a_1)\cdots t^{\ell_m}(a_m)(T^r(\mathsf{c})) \\ = \;& a_{m+1}\cdots a_{2n} G^d t^{\ell_1}(a_1)\cdots t^{\ell_m}(a_m)(\mathsf{c}) \\ = \;& G^d t^{\ell_1}(a_1)\cdots t^{\ell_m}(a_m)(a_{m+1}\cdots a_{2n})(\mathsf{c}) \end{aligned}$$

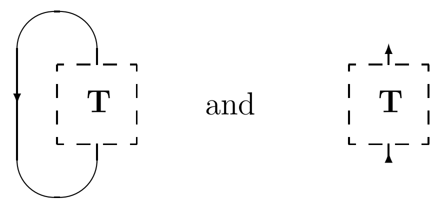
which is $a_1\otimes\cdots\otimes a_{2n}$ applied to the left hand side of the equation of (16). We have established (16).

Set

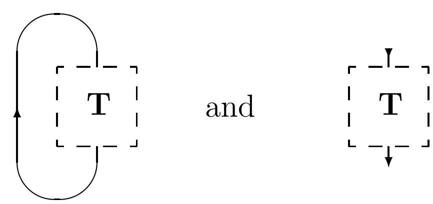
$$\mathbf{f}_{C,\mathsf{c}}(\mathbf{K}) = (G^d\mathbf{f})(\mathsf{c}). \tag{17}$$

The preceding calculations show that (17) describes a well-defined function on $\mathbf{K}$, which by abuse of notation we will refer to as $\mathbf{f}_{C,\mathsf{c}} : \mathcal{K} \longrightarrow k$.

Observe that the oriented knot diagram $\mathbf{K}$ is regularly isotopic to an oriented knot diagram $\mathbf{K}(\mathbf{T})$, where $\mathbf{K}(\mathbf{T})$ is



is an oriented 1–1 tangle diagram, or $\mathbf{K}(\mathbf{T})$ is



is an oriented 1–1 tangle diagram. Since the Whitney degree is a regular isotopy invariant of oriented knot diagrams, the Whitney degrees of $\mathbf{K}$ and $\mathbf{K}(\mathbf{T})$ are the same.

Theorem 6 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}}, G)$ be a twist oriented quantum coalgebra over the field k, let c be a $T_{\mathsf{d}} \circ T_{\mathsf{u}}$-invariant cocommutative element of C, and let $\mathbf{f}_{C,\mathsf{c}} : \mathcal{K} \longrightarrow k$ be the function defined by (17).*

a) *Suppose that $\mathbf{K}, \mathbf{K}' \in \mathcal{K}$ are regularly isotopic. Then $\mathbf{f}_{C,\mathsf{c}}(\mathbf{K}) = \mathbf{f}_{C,\mathsf{c}}(\mathbf{K}')$.*

b) *Suppose that $\mathbf{K} \in \mathcal{K}$ and that $\mathbf{K}$ is regularly isotopic to $\mathbf{K}(\mathbf{T})$ for some $\mathbf{T} \in \mathbf{Tang}$. Then*
$$\mathbf{f}_{C,\mathsf{c}}(\mathbf{K}) = (G^d \mathbf{T}_C)(\mathsf{c}),$$
where d is the Whitney degree of $\mathbf{K}$.

c) $\mathbf{f}_{C,\mathsf{c}}(\mathbf{K}^{op}) = \mathbf{f}_{C^{cop},\mathsf{c}}(\mathbf{K})$ *for all $\mathbf{K} \in \mathcal{K}_{\text{knots}}$.*

$\square$

Observe that the formula in part b) of the preceding theorem may be written

$$\mathbf{f}_{C,\mathsf{c}}(\mathbf{K}) = (G^d \mathbf{T}_C)(\mathsf{c}) = \mathbf{T}_C(G^d(\mathsf{c}_{(1)})\mathsf{c}_{(2)}) = \mathbf{T}_C(\mathsf{c} \leftharpoonup G^d).$$

Part c) of the preceding theorem follows with this observation together with the fact that we may assume $\mathbf{K} = \mathbf{K}(\mathbf{T})$ for some $\mathbf{T} \in \mathbf{Tang}^o_{\text{tangles}}$.

Observe that $\mathbf{f}_{C,\mathsf{c}}(\mathbf{K}) = G^d(\mathsf{c})$ when $\mathbf{K}$ has no crossings.

8.2 The function $\mathbf{f}_{C,\mathsf{c}}$ Defined for Oriented Link Diagrams

Let $\mathbf{L} \in \mathcal{L}$ be an oriented link diagram with components $\mathbf{L}_1, \ldots, \mathbf{L}_r$ and suppose that c a $T_\mathsf{d}\circ T_\mathsf{u}$-invariant cocommutative element of the twist oriented quantum coalgebra C. To construct the scalar $\mathbf{f}_{C,\mathsf{c}}(\mathbf{L})$ we modify the procedure for the construction of $\mathbf{f}_{C,\mathsf{c}}(\mathbf{K})$, where $\mathbf{K} \in \mathcal{K}$ is an oriented knot diagram, described in the preceding section.

For each $1 \leq \ell \leq r$ let d_ℓ denote the Whitney degree of the component $\mathbf{L}_\ell$, let

$$\mathsf{c}(\ell) = \mathsf{c}\leftharpoonup G^{d_\ell} = G^{d_\ell}(\mathsf{c}_{(1)})\mathsf{c}_{(2)}$$

and choose a point on a vertical line of $\mathbf{L}_\ell$. We refer to this point as a starting point. (As in the case of knot diagrams we can always assume that each component of $\mathbf{L}$ has a vertical line.) Traverse the component $\mathbf{L}_\ell$, beginning at the starting point and moving in the direction of the orientation, labelling the crossing lines contained in $\mathbf{L}_\ell$ by $(\ell{:}1), (\ell{:}2), \ldots$ in the order encountered. Let $u(\ell{:}\imath)$ denote the number of local extrema which are traversed in the counterclockwise direction minus the number of local extrema which are traversed in the clockwise direction during the portion of traversal of the link component from line labelled $(\ell{:}\imath)$ to the starting point.

Next we construct a scalar $\mathbf{f}'_{C,\mathsf{c}}(\mathbf{L})$. If $\mathbf{L}$ has no crossings we set $\mathbf{f}'_{C,\mathsf{c}}(\mathbf{L}) = 1$. Suppose that $\mathbf{L}$ has at least one crossing. Then we define $\mathbf{f}'_{C,\mathsf{c}}(\mathbf{L})$ to be a sum of products, where each crossing contributes a factor of the form

$$\cdots b^{\pm}(T_\mathsf{d}^\bullet\circ T_\mathsf{u}^\bullet(\bullet), T_\mathsf{d}^\bullet\circ T_\mathsf{u}^\bullet(\bullet))\cdots$$

according to the conventions of Section 6.1, where $(\ell{:}\imath)$ replaces $\imath$, $(\ell'{:}\imath')$ replaces $\jmath$, and then $\mathsf{c}(\ell)_{(\imath)}$ replaces $c_{(\ell:\imath)}$ and $\mathsf{c}(\ell')_{(\imath')}$ replaces $c_{(\ell':\imath')}$.

We define

$$\mathbf{f}_{C,\mathsf{c}}(\mathbf{L}) = \omega\mathbf{f}'_{C,\mathsf{c}}(\mathbf{L}), \tag{18}$$

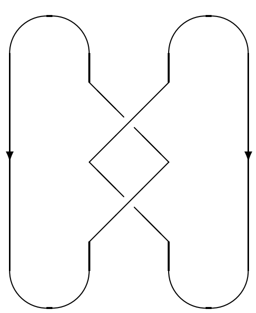
where ω is the product of the $G^{d_\ell}(\mathsf{c})$'s such that the component $\mathbf{L}_\ell$ has no crossing lines. The reader is left with the exercise of showing that $\mathbf{f}_{C,\mathsf{c}}(\mathbf{L})$ does not depend on the particular starting points and is not affected by the replacement of local parts of the diagram $\mathbf{L}$ by their equivalents according to (M.1)–(M.5) and (M.2rev)–(M.5rev). The proof of Theorem 4 provides a blueprint for the latter. Collecting results:

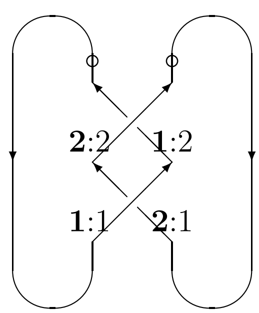
Theorem 7 *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}}, G)$ be a twist oriented quantum coalgebra over the field k, suppose that c is a $T_{\mathsf{d}} \circ T_{\mathsf{u}}$-invariant cocommutative element of C and let $\mathbf{f}_{C,\mathsf{c}} : \mathcal{L} \longrightarrow k$ be the function described by (18). If $\mathbf{L}, \mathbf{L}' \in \mathcal{L}$ are regularly isotopic then $\mathbf{f}_{C,\mathsf{c}}(\mathbf{L}) = \mathbf{f}_{C,\mathsf{c}}(\mathbf{L}')$.*

$\square$

Observe that $\mathbf{f}_{C,\mathsf{c}}$ restricted to $\mathcal{K}$ is the function described in (17). By virtue of the preceding theorem the function $\mathbf{f}_{C,\mathsf{c}}$ determines a regularly isotopy invariant of oriented links. When C is the dual twist quantum oriented coalgebra of a finite-dimensional twist oriented quantum algebra A over k then the scalar $\mathbf{f}_{C,\mathsf{c}}(\mathbf{L})$ is the invariant $K(L)$ of [6] defined for A. See also [3].

We end this section with two examples, the Hopf link and the Borromean rings. Consider the oriented Hopf link $\mathbf{L}_{Hopf}$ depicted below left with components $\mathbf{L}_1$ and $\mathbf{L}_2$, reading left to right. The symbol $\circ$ denotes a starting point.



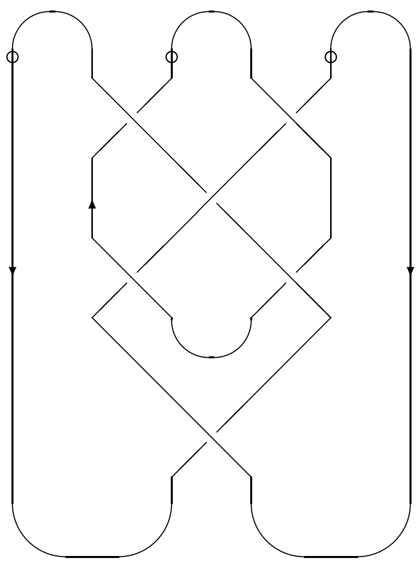
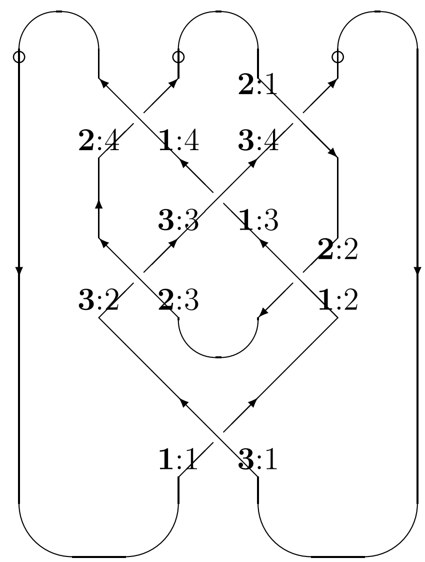


Observe that $d_1 = -1$, $d_2 = 1$ and

$$\mathbf{f}_{C,\mathsf{c}}(\mathbf{L}_{Hopf}) = b(d_{(1)}, e_{(1)})b(e_{(2)}, d_{(2)}),$$

where $d = c \leftharpoonup G^{-1}$ and $e = c \leftharpoonup G$.

Suppose $\mathbf{L}_{\text{Borro}}$ is the Borromean rings with the orientation given in the diagram below left and let $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$ be the components of $\mathbf{L}_{\text{Borro}}$, reading from left to right.

Observe that $d_1 = -1$, $d_2 = 1 = d_3$ and

$$\begin{aligned}\mathbf{f}_{C,\mathsf{c}}(\mathbf{L}_{\mathrm{Borro}}) = b^{-1}(e_{(1)}, c_{(1)})b^{-1}(T^2(c_{(2)}), d_{(2)})b(e_{(3)}, c_{(3)})\times \\ b^{-1}(c_{(4)}, d_{(4)})b^{-1}(d_{(3)}, e_{(2)})b^{-1}(d_{(1)}, e_{(4)})\end{aligned}$$

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